PART C - MATHS

SECTION I - MULTIPLE ANSWER CORRECT TYPE

1. a) The normals is $y^2 = 4ax$ at A, B, C meet at a point

b) The circumcircle of ΔABC passes through the origin

c) The centroid of ΔABC lies on the x-axis.

a) coefficient of $t^2 = 0 \Rightarrow t_1 + t_2 + t_3 = 0$

$\Rightarrow$ A, B, C are co-normal points

$\Rightarrow$ three normals at A, B, C meet at a point.

b) Let the circumcircle pass through D(t_4).

Let the circumcircle's equation be

$x^2 + y^2 + 2gx + 2fy + c = 0$. $x = at^2$, $y = 2at$ give

$a^2t^4 + (4a^2 + 2ag)t^2 + 4aft + c = 0$

Coefficient (t^3) = 0.

$\Rightarrow t_1 + t_2 + t_3 + t_4 = 0 \qquad \Rightarrow t_4 = 0$

$\Rightarrow$ D is nothing but the origin.

c) The axis of the parabola is x-axis ($y = 0$)

Centroid of ΔABC

$$= \left(\frac{a(t_1^2 + t_2^2 + t_3^2)}{3}, \frac{2a(t_1 + t_2 + t_3)}{3}\right)$$

$$= \left(\frac{a(t_1^2 + t_2^2 + t_3^2)}{3}, 0\right)$$

$\Rightarrow$ Centroid lies on the x-axis.

d) Slope of chord AB $= \dfrac{2a(t_2 - t_1)}{a(t_2^2 - t_1^2)} = \dfrac{2}{t_1 + t_2} = \dfrac{-2}{t_3}$

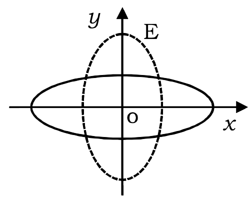
& slope of tangent at C $= \dfrac{2a}{2at_3} = \dfrac{1}{t_3}$

$$\frac{-2}{t_3} + \frac{1}{t_3} \neq 0$$

2. b) If the tangents from P to given ellipse are mutually perpendicular then tangent from P to ellipse E are also mutually perpendicular

c) The distance between a focus of the given ellipse and a focus of ellipse E is $\sqrt{14}$

d) common tangent to two ellipse have slopes ± 1



a) Auxiliary circle of ellipse E is $x^2 + y^2 = 16$

b) P lies on director circle to given ellipse.

Director circle of given ellipse and ellipse E is same i.e. $x^2 + y^2 = 16 + 9 = 25$

c) Eccentricity of given ellipse $= \sqrt{1 - \dfrac{9}{16}} = \dfrac{\sqrt{7}}{4}$

Focus of given ellipse = $(\pm\sqrt{7}, 0)$

Focus of ellipse E = $(0, \pm\sqrt{7})$

Distance between one focus of given ellipse and that of ellipse E = $\sqrt{7+7} = \sqrt{14}$

Let m be the slope of common tangent. Then

$9m^2 + 16 = 16m^2 + 9$

$\Rightarrow 7m^2 = 7 \qquad \Rightarrow m = \pm 1$

3. a) –7

Let P′(h, k) be the reflection(image) of P(α, β) in $y = 2x$. Let Q be the mid point of PP′. Then Q = $\left(\dfrac{\alpha + h}{2}, \dfrac{\beta + k}{2}\right)$. Q lies on $y = 2x$

$\Rightarrow \beta + k = 2(\alpha + h)$...(1) & $h + 2k = \alpha + 2\beta$ ...(2)

(1) & (2) $\Rightarrow \alpha = \dfrac{4k - 3h}{5}$, $\beta = \dfrac{4h + 3k}{5}$

$\alpha\beta = 1 \Rightarrow 12k^2 + 7kh - 12h^2 = 25$

$\Rightarrow$ (h, k) satisfies $12x^2 - 7xy - 12y^2 + 25 = 0$

$r = -7$

4. b) (–1, 1)

The equation of circle is

$x^2 - y^2 - a^2 + \lambda(x^2 - y) = 0$ ...(1)

$\Leftrightarrow (1+\lambda)x^2 - y^2 - a^2 - \lambda y = 0$ ...(2)

For (2) to represent a circle $\lambda + 1 = -1$

$\Rightarrow \lambda = -2$

$\therefore$ (1) $\Leftrightarrow x^2 + y^2 - 2y + a^2 = 0$

$\Leftrightarrow x^2 + (y - 1)^2 = -a^2 + 1$

$\Rightarrow 1 - a^2 > 0 \qquad \Rightarrow a^2 < 1$

$\Rightarrow -1 < a < 1$

5. b) $x^2 = \frac{4}{3}(y-3)$

Let P = $(2at_1, at_1^2)$, Q = $(2at_2, at_2^2)$, $a = 1$

OP ⊥ OQ $\Rightarrow t_1t_2 = -4$ ...(1)

Let (h, k) be the centriod of ΔPFQ

$h = \frac{2}{3}(t_1 + t_2)$ & $k = \frac{1}{3}(t_1^2 + t_2^2 - 1)$ ...(3)

(1), (2), (3) $\Rightarrow \frac{9h^2}{4} = 3(k-3)$

$\Rightarrow$ (h, k) satisfies $3x^2 = 4(y-3)$

6. b) Equation of the directrix is $4x - 3y - 1 = 0$

c) Length of the latus rectum is 10 units

d) Equation of latus rectum is $4x - 3y + 24 = 0$

Let M be foot of the directrix & F be the focus. Then slope of the axis of parabola = slope of FM

a) Slope of the axis = $\frac{1-4}{1+3} = -\frac{3}{4}$

b) Slope of the directrix $= \frac{4}{3}$, Directrix passes through(1, 1)

$\Rightarrow$ Equation of the directrix : $4x - 3y - 1 = 0$

c) $|MF| = 2a = \sqrt{(-3-1)^2 + (4-1)^2} = 5$

$\Rightarrow$ Length of the latus rectum = 10

d) Latus rectum is parallel to the directrix.

Equation of latus rectum : $4x - 3y + \lambda = 0$

(λ : a constant)

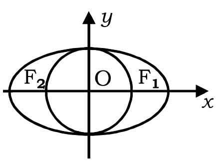
Latus rectum passes through F(– 3, 4).

$\Rightarrow \lambda = 24$

7. a) Eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$.

c) The ratio of the square of radius of director circle to the square of the radius of circle ω is 3 : 1.

d) The circle ω and auxiliary circle are concentric.



$2ae = 2b \qquad \Rightarrow b = ae$

$b^2 = a^2(1 - e^2) \qquad \Rightarrow 2b^2 = a^2$

$\therefore \frac{1}{2} = 1 - e^2 \qquad \Rightarrow e = \frac{1}{\sqrt{2}}$

The radius of the circle ω is $ae = \frac{a}{\sqrt{2}}$

The square of radius of director circle $= a^2 + b^2 = 3b^2$

$\Rightarrow$ The ratio of square of radii of director circle and circle ω is 3 : 1

8. a) The curve C is a hyperbola.

b) Centre of curve C is (3, 2).

c) Eccentricity of curve C is $\frac{4}{3}$.

d) Length of latus rectum is $\frac{14}{3}$.

Given equation $\Leftrightarrow 7(y-2)^2 - 9(x-3)^2 = 63$

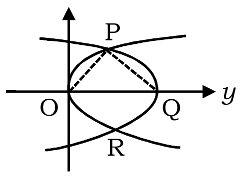
$\Leftrightarrow \frac{(y-2)^2}{9} - \frac{(x-3)^2}{7} = 1$. Equation represents a hyperbola ($a^2 = 7$, $b^2 = 9$). Centre of the hyperbola = (3, 2) (Taking X = $x - 3$, Y = $y - 3$ and $x = 0$, $y = 0$)

$a^2 = b^2(e^2 - 1) \qquad \Rightarrow e = \frac{4}{3}$

length of latus rectum $= \frac{2a^2}{b} = \frac{14}{3}$

SECTION II – Paragraph Type

9. d) $8\sqrt{2}$



Two parabolae meet at P and R where

$P = \left(\frac{2a}{3}, \sqrt{\frac{8a}{3}}\right)$ and $R = \left(\frac{2a}{3}, -\sqrt{\frac{8a}{3}}\right)$

OPQR is cyclic quadrilateral with

$\angle OPQ = 90^0 \Rightarrow \dfrac{\sqrt{\frac{8a}{3}}}{\frac{2a}{3}} \times \dfrac{\sqrt{\frac{8a}{3}}}{\frac{2a}{3} - a} = -1$

$\Rightarrow \dfrac{8a/3}{2a^2/9} = 1 \qquad \Rightarrow a = 12$

$\Rightarrow$ |PR| = length of common chord = $2\sqrt{32}$

$= 8\sqrt{2}$

10. b) $48\sqrt{2}$

Area of quadrilateral OPQR

$= \frac{1}{2} |OQ| \times |PR| = \frac{1}{2} \times 12 \times 8\sqrt{2} = 48\sqrt{2}$

SECTION III (Matrix Match Type)

1. A-S, B-R, C-Q, D-P

$|P_1F_1| . |P_2F_2|$ = (semi minor axis)^2 = 3

(A) → (s)

The fixed point (p, q) is focus of the parabola

$(p, q) = (2, 0)$ **(B) → (r)**

$(4, 4) = (at_1^2, 2at_1)$, $a = 1$

$\Rightarrow t_1 = 2$ & $(1, -2) = (at_2^2, 2at_2)$ $\Rightarrow t_2 = -1$

Point of intersection is $(at_1t_2, a(t_1 + t_2))$

$= (-2, 1) = (p, q)$

$\Rightarrow p + 3q = 1$ **(C) → (q)**

Eccentricity of a rectangular hyperbola

$e_1 - e_2 = 0$ **(D) → (p)**

SECTION IV - INTEGER TYPE

1. 2

Slope of tangent $m = \frac{1}{4}$, $a = \frac{7}{4}$

point of contact $= \left(\frac{a}{m^2}, \frac{2a}{m}\right) = (28, 14)$

2. 4

$y^2 = kx - 8 = k\left(x - \frac{8}{k}\right)$ which is of the

form $Y^2 = 4aX$ where $a = \frac{k}{4}$, $X = x - \frac{8}{k}$

Directrix $X = -a \quad \Leftrightarrow x - \frac{8}{k} = -\frac{k}{4}$

Also $x = 1 \quad \Rightarrow \frac{8}{k} - \frac{k}{4} = 1$

$\Leftrightarrow k^2 + 4k - 32 = 0$

$\Leftrightarrow (k + 8)(k - 4) = 0$

$\Leftrightarrow k = 4, -8$

3. 6

Given equation

$\Leftrightarrow (x - 2)^2 + (y + 3)^2 = \frac{1}{5} . \frac{(x - 2y + 7)^2}{5}$

$\Rightarrow$ eccentricity $= \frac{1}{\sqrt{5}}$,

directrix is $x - 2y + 7 = 0$ and one of the foci is (2, –3). Distance between focus and the directrix is $\frac{a}{e} - ae$.

$a\left(\sqrt{5} - \frac{1}{\sqrt{5}}\right) = \frac{|2 + 6 + 7|}{\sqrt{5}}$

$\Rightarrow 4a = 15$

length of major axis $= k = 2a = \frac{15}{2}$

$\Rightarrow \frac{4}{5} k = 6$

4. 7

Length of the intercept

$= \sqrt{16\sec^2\theta + 9\text{cosec}^2\theta}$

$= \sqrt{25 + 16\tan^2\theta + 9\cot^2\theta}$

$16\tan^2\theta + 9\cot^2\theta \ge \underbrace{2(4\tan\theta)(3\cot\theta)}_{=24}$

$\therefore$ length of the intercept $\ge \sqrt{25 + 24} = 7$

5. 1

Equation of chord is $\frac{x}{\sqrt{3}} - \frac{y}{2\sqrt{3}} = \frac{3}{2}$

(use mid-point)

$\Leftrightarrow 2x - y = 3\sqrt{3}$

Solve it with equation of hyberbola to get

$3x^2 - 12\sqrt{3}x + 33 = 0$

$\Leftrightarrow x^2 - 4\sqrt{3} + 11 = 0 \qquad ...(1)$

$(1) = x_1 + x_2 = 4\sqrt{3}$, $x_1x_2 = 11 \quad ...(2)$

Also, $2x_1 - y_1 = 3\sqrt{3}$ & $2x_2 - y_2 = 3\sqrt{3}$

$\Rightarrow 2(x_1 - x_2) = (y_1 - y_2)$

Length of the chord

$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{5}\ \sqrt{(x_1 - x_2)^2}$

$= \sqrt{5}\ \sqrt{48 - 4(11)}$

$= 2\sqrt{5}$

6. 1

The asymptotes are $x - y = 0$, $x + y = 0$

The required product = $\dfrac{|x_1 - y_1|}{\sqrt{2}} \cdot \dfrac{|x_1 + y_1|}{\sqrt{2}}$

$= \dfrac{x_1^2 - y_1^2}{2} = \dfrac{2}{2} = 1$

7. 3

The foot of the perpendicular from focus to a tangent of given hyperbola lies on auxiliary circle $x^2 + y^2 = 9$.

8. 5

Substitute $t = \dfrac{y/3}{-\frac{x}{4} + 1}$ in $\dfrac{tx}{4} - \dfrac{y}{3} + t = 0$ to

get $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

Radius = $\sqrt{16 + 9}$ = 5