LAKSHYA ADVANCED UNIT TEST (LAUT)

00 – 00		Q. Booklet Serial No: 151115	
Test No : 2107	3 Hrs.		

Hints & Solutions

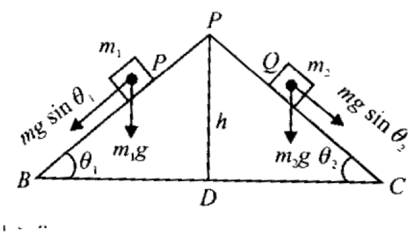
PART A - PHYSICS

SECTION I - MULTIPLE ANSWER CORRECT TYPE

1. **a) The density continuously increases from left to right.**
   b) The density continuously decreases from left to right.
   Distribution is uniformly is uniformly uneven. Heavier part will have the CM closer than the lighter part.

2. **a) may not move**
   b) must not accelerate
   c) may move
   In the absence of external forces : $a_{CM} = 0$
   $\left(\because \sum F_{ext} = Ma_{CM}\right)$
   So CM will not accelerate. But if the system was moving initially, it will continue to move with same velocity.

3. **b) block Q will reach the bottom earlier than block B**
   c) both blocks will reach the bottom with the same speed
   The acceleration of blocks P and Q are
   $a_1 = \frac{m_1 g \sin\theta_1}{m_1} = g \sin\theta_1$ and
   $a_2 = \frac{m_2 g \sin\theta_2}{m_2} = g \sin\theta_2$



   Since $\theta_2 > \theta_1$; $a_2 > a_1$.
   Now PE of block P at A = m_1gh
   Its KE on reaching the bottom = $\frac{1}{2} m_1 v_1^2$
   Equating both, we get
   $\frac{1}{2} m_1 v_1^2 = m_1 gh$ or $v_1 = \sqrt{2gh}$
   Similarly, for block Q $v_2 = \sqrt{2gh}$
   Since $v_1 = v_2$, both blocks will reach the bottom with the same
   speed. Now, $v_1 = a_1 t_1$ (u = 0) and $v_2 = a_2 t_2$.
   But $v_1 = v_2$
   Therefore, $a_1 t_1 = a_2 t_2$
   $\frac{t_1}{t_2} = \frac{a_2}{a_1}$
   Since, $a_2 > a_1$, $t_1 > t_2$, i.e., block P takes a longer time to reach the bottom. Hence, the correct choices are b. and c.

4. **b)** $\delta = \frac{2F}{k}$
   c) work done by force F is equal to Fδ
   Let mass of the block hanging from the spring be m. Then initial elongation of the spring will be equal to mg/k. When the force F is applied to pull the block down, the work done by is used to increase strain energy of this spring.
   $(F\delta = mg\delta) = \left[\frac{1}{2}k\left(\frac{mg}{k} + \delta\right)^2 - \frac{m^2g^2}{2k}\right]$
   From this Eq., $\delta = 2F/k$
   Hence, option **b**. is correct.
   Since, a constant force is applied on the block to pull it down, therefore during this process, work done by force, $W = F\delta$.
   Hence, option **c**. is also correct.

Increase in energy of the spring is equal to

$$\left[\frac{1}{2}k\left(\frac{mg}{k}+\delta\right)^2 - \frac{m^2g^2}{2k}\right]$$

From this equation, increase in energy is not equal to $\frac{1}{2}k\delta^2$.

Hence, option **d**. is wrong.

5. **a) Loss in gravitational potential energy is 10 J**
   b) Kinetic energy of 1kg block is 0.045 J
   c) 4kg block travels a distance of 2m to acquire a velocity of 0.6 ms^{-1}
   d) Work done against friction is 80μ J where μ is coefficient of kinetic friction

a. Loss of GPE = mgh
$= 1.0 \times 10 \times 1 = 10$ J

b. KE of 1 kg block = $(1/2)$ mv^2
$= (1/2) \times 1 \times (0.3)^2 = 0.045$ J

c. Using string constraint

d. Work done against friction = μmgs
$= \mu \times 4.0 \times 10 \times 2 = 80\mu$ J

6. **a) $p = \sqrt{2mK}$**
   c) $2K = p\upsilon$

$p = mv$ or $p^2 = m^2v^2$, or

$$\frac{p^2}{2m} = \frac{1}{2}mv^2 = K$$

$p = \sqrt{2mK}$

Also, $p = mv$ and $K = \frac{1}{2}mv^2$

Dividing the two, we get $2K = pv$

Hence, the correct relations are **a**. and **c**.

7. **c) the forces acting on them are equal and opposite**
   d) their momenta are equal and opposite

When two blocks connected by a spring move towards each other, law of conservation of momentum leads to the following results :

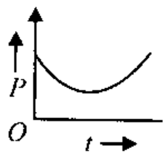
i. $\vec{p}_2 = -\vec{p}_1$. i.e., at any instant two blocks will have equal and opposite momenta.

ii. $\vec{v}_2 = -\left(\frac{m_1}{m_2}\right)\vec{v}_1$ i.e., two blocks move in opposite directions with lighter block moving faster.

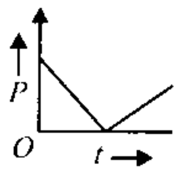
iii. As $K = \frac{p^2}{2m}$, Therefore, $\frac{K_1}{K_2} = \frac{m_2}{m_1}$ i.e. KE of the two blocks will not be equal; the lighter block will have greater kinetic energy.

iv. It is quite clear that forces acting on them will be equal and opposite. So, options **c**. and **d**. are correct.

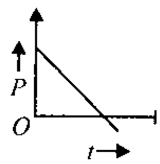
8. **a)**



b)



c)



Rate of work done is the power associated with the force. It means rate of work done by the gravitational force is the power associated with the gravitational force.

Gravitational force acting on the block is equal to its weight mg which acts vertically downwards.

Velocity of the particle, at time t, has two components :

a. horizontal component $v\cos\theta$ and **b.** vertically upward component $(v_0 \sin\theta - gt)$.

Hence, the power associated with the weight mg will be equal to

$P = m\vec{g}.\vec{v} = -mg(v_0 \sin\theta - gt)$

This shows that the curve between power and time will be a straight line having positive slope but negative intercept on y-axis. Hence, only option **d**. is correct, i.e., rest of the options **a**,. **b**., **c**. are not correct.

SECTION II - MATRIX MATCH TYPE

1. **A– Q, B – P, C – R, D–S**

Q. Initial velocity of centre of mass of given system is zero and net external force is in vertical direction. Since there is shift of mass downwards, the centre of mass has only downward shift.

P. Obviously, there is shift of centre of mass of given system downwards. Also the pulley exerts a force on string which has a horizontal component towards right. Hence, centre of mass of system has rightward shift.

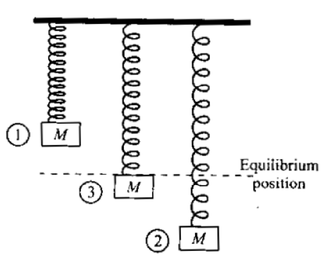
R. Both block and monkey move up, hence centre of mass of the given system shifts vertically upwards.

S. Net external force on given system is zero. Hence, centre of mass of the given system remains at rest.

2. **A– P ; B –R,S ; C – P,S ; D – Q,S**

In the figure, 3 is the equilibrium position where velocity is maximum and acceleration is zero. 1 and 2 are the extreme positions where velocity is zero and acceleration is maximum. I is the unstretched position.

When the block is at position '3', then mg = kx. So net force is zero, hence acceleration is zero. But velocity may be either in upward or downward direction.=



When the block is between position '3' and '2', then kx > mg. So the net force is in upward direction, hence acceleration is in upward direction. But the velocity may be either in upward or downward direction.

But if the block is at position '2', then the velocity is zero and the acceleration in upward direction.

When the block is between position '3' and '1', then mg > kx. So the net force in downward direction, hence acceleration is in downward direction. But velocity may be either in upward or downward direction.

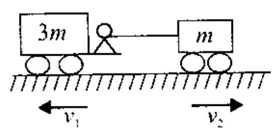
SECTION III - INTEGER TYPE

1. **9**

Trolleys gain momentum due to force applied by man which will be internal force for the system of trolleys and man and there is no other external force. Here we assume that man applies force for a very short time, during which effect of friction can be neglected.

Momentum just before pushing = momentum just after pushing.

$0 = 3\,mv_1 - mv_2 \quad v_1 = \frac{v_2}{3}$



From work-energy theorem for individual trolleys,

$f_1S_1 = \frac{1}{2}3mv_1^2,\ f_2S_2 = \frac{1}{2}mv_2^2$

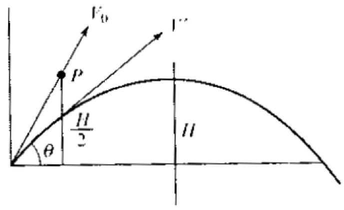
Here $f_1 = \mu\,3mg,\ g_2 = \mu\,mg$

Solve to get $\frac{S_2}{S_1} = \left(\frac{v_2}{v_1}\right)^2 = 9$

2. **3**

From energy conservation = $\frac{1}{2}mv_0^2\cos^2\theta$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2\cos^2\theta\frac{mgH}{2}$$



$v_0 \cos\theta = \sqrt{\frac{2}{5}}$

From (i) and (ii), $\theta = 60^0 = \pi/3$.

3. **0**

As $F_1 = F_2 \cos 60^0$, the block will move with constant velocity. Power due to force $\vec{F_1}$ is

$P_1 = \vec{F_1}.\vec{v} = -2.0 \times 3.0$ W $= -6.0$ W

$P_2 = \vec{F_2}.\vec{v} = +4.0 \times 3.0 \cos60^0$ W $= 6.0$ W

4. **7**

Loss in K.E. = work done against the retarding force F offered by the target

$$\frac{1}{2}mv^2 - \frac{1}{2}m\left(\frac{v}{2}\right)^2 = Fx$$

and $\frac{1}{2}mv^2 - 0 = Fx'$

From (i) and (ii),

$$\frac{x'}{x} = \frac{4}{3}$$

$$x' = \frac{4}{3}.x = \frac{4}{3}(21) = 28\,cm$$

$(x' - x) = 7$ cm

5. **6**

$(1 + 3)$ v $= (1)(8) + (3)(4) = 20$

v = 5 m/sec

For block A, $W_f = \frac{1}{2}(1)(5^2 - 8^2) = -\frac{39}{2}J$

For block B, $W_f = \frac{1}{2}(3)(5^2 - 4^2) = +\frac{27}{2}J$

Net work done by friction = – 6 J

6. **4**

As the centre of gravity of the rope comes up by h/2 and that of the bucket by h, the required work is

$W = \frac{mgh}{2} + Mgh = 4{,}000$ J = 4 kJ

7. **8**

Mass of water flowing out per second is

$m = Av\rho$

Rate of increase of kinetic energy

$$= \frac{1}{2}mv^2 = \frac{1}{2}A\rho^3$$

$$\frac{P'}{P} = \frac{(A\rho v'^3)}{(A\rho v^3)} = \frac{v'^3}{v^3}$$

8. **3**

The change in the position of the centre of mass is given by

$$\Delta x_{cm} = \frac{m_1x - m_2y}{m_1 + m_2}$$

The negative sign for the second is because we have taken the displacement of m_2 is towards left. As the shift in the centre of mass is zero, $\Delta x_{cm} = 0$.

$\Rightarrow m_1x - m_2y = 0 \;\; \Rightarrow y = \frac{m_1}{m_2}x$

$\therefore y = \frac{5}{3} \times 1.8 = 3$ m to the left

PART C - MATHS

SECTION I - MULTIPLE ANSWER CORRECT TYPE

1. **a)** $\boldsymbol{x - 2y = 0}$
   c) $\boldsymbol{x - 2y = \pi}$
   d) $\boldsymbol{x - 2y + \pi = 0}$

$\tan^2(a + 2)b + a^2 = 0$

$a = 0, \tan^2(a + 2)b = 0$

$\Rightarrow \tan^2 2b = 0$

$\Rightarrow b = 0, \frac{\pi}{2}, -\frac{\pi}{2}$

$(a, b) = (0, 0), \left(0, \frac{\pi}{2}\right), \left(0, -\frac{\pi}{2}\right)$

$y - 0 = \frac{1}{2}(x - 0), \quad y - \frac{\pi}{2} = \frac{1}{2}(x - 0)$

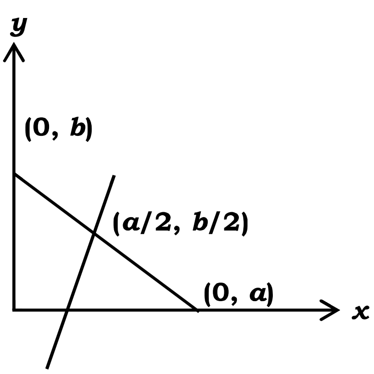
$y + \frac{\pi}{2} = \frac{1}{2}(x - 0)$

$2y = x, 2y - \pi = x, 2y + \pi = x$

2. **a)** $\boldsymbol{\left(\frac{a+b}{2}, \frac{a+b}{2}\right)}$
   c) $\boldsymbol{\left(\frac{a-b}{2}, \frac{b-a}{2}\right)}$

$$\frac{x - \frac{a}{2}}{\frac{b}{\sqrt{a^2 + b^2}}} = \frac{y - \frac{b}{2}}{\frac{a}{\sqrt{a^2 + b^2}}} = \pm\frac{\sqrt{a^2 + b^2}}{2}$$

$x = \frac{a}{2} + \frac{b}{2}, y = \frac{b}{2} + \frac{a}{2}$ and $x = \frac{a}{2} - \frac{b}{2}, y = \frac{b}{2} - \frac{a}{2}$



$\therefore$ The required points are

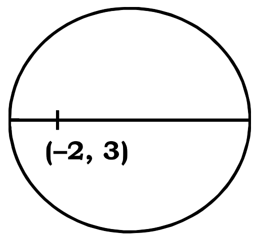
$\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ and $\left(\frac{a-b}{2}, \frac{b-a}{2}\right)$.

3. **a)** $\boldsymbol{a + b = 18}$
   c) $\boldsymbol{a - b = 4\sqrt{2}}$
   d) $\boldsymbol{a - b = 73}$

$x^2 + y^2 + 8x - 10y - 40 = 0$

Centre of the circle is (–4, 5)

Its radius = 9



Distance of the centre (–4, 5) from the point (–2, 3) is $\sqrt{4+4} = 2\sqrt{2}$

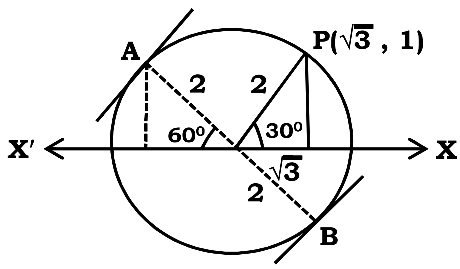
$\therefore a = 2\sqrt{2} + 9$ and $b = -2\sqrt{2} + 9$

$\therefore a + b = 18$

$a - b = 4\sqrt{2}$

$a.\ b = 81 - 8 = 73$

4. **b)** $\sqrt{3}y = x + 4$
   d) $\sqrt{3}y = x - 4$



Clearly, A ≡ (–2 cos 60°, 2 sin 60°) and
B ≡ (2 cos 60°, –2 sin 60°).

The tangent at A is
$x(-2 \cos 60^\circ) + y(2 \sin 60^\circ) = 4$ and that at B is $x(2 \cos 60^\circ) + y(-2 \sin 60^\circ) = 4$.

5. **a)** $\boldsymbol{\sec 2\alpha = h}$
   b) $\boldsymbol{\cos\alpha = \sqrt{(1 + h)/(2h)}}$
   d) $\boldsymbol{\cot\alpha = \sqrt{(h + 1)/(h - 1)}}$

Let the equation of the lines given by $x^2 + 2hxy + y^2 = 0$ be $y = m_1x$. Since these make an angle α with $y + x = 0$ whose slope is -1, we have

$$\frac{m_1+1}{1-m_1} = \tan\alpha \frac{-1-m_2}{1-m_2}$$

or $m_1 + m_2 = \dfrac{(\tan\alpha-1)^2+(\tan\alpha+1)^2}{\tan^2\alpha-1}$

$= \dfrac{-2\sec^2\alpha\times\cos^2\alpha}{\cos 2\alpha}$

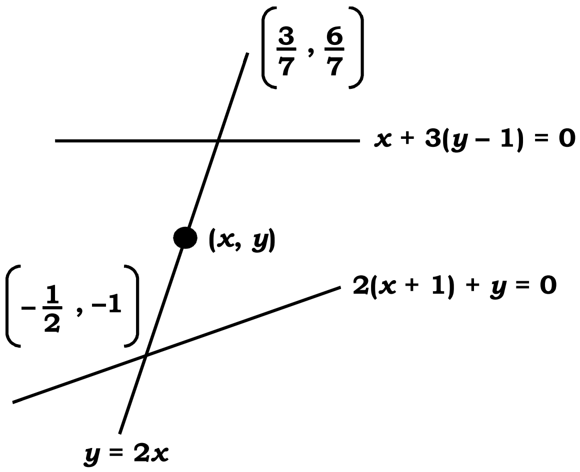
$\therefore$ $-2\sec 2\alpha = -2h$

or $\sec 2\alpha = h$

or $\cos 2\alpha = \dfrac{1}{h}$ or $2\cos^2\alpha - 1 = \dfrac{1}{h}$

or $\cos\alpha = \sqrt{\dfrac{1+h}{2h}}$ and $\cot\alpha = \sqrt{\dfrac{h+1}{h-1}}$

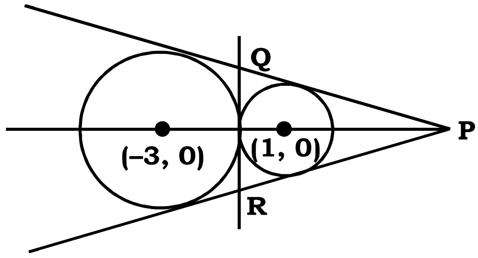
6. **b)** $\boldsymbol{x \in (-1/2, 3/7)}$
   d) $\boldsymbol{y \in (-1, 6/7)}$



Solving $y = 2x$, $2(x + 1) + y = 0$, we get $x = -1/2$, $y = -1$.
Solving $y = 2x$, $x + 3(y - 1) = 0$, we get $x = 3/7$, $y = 6/7$.

7.

Centre of the first circle is $(-3, 0)$ and the radius is 3 and radius of the second circle is $(1, 0)$ and the radius is 1. Since, the distance between the centres is equal to the sum of the radii, the two circles touch each other externally at the origin, the common tangent at the origin is y-axis.
Let $y = mx + c$ be a direct common tangent to the two circles, then



$$\frac{-3m+c}{\sqrt{1+m^2}} = \pm 3 \text{ and } \frac{m+c}{\sqrt{1+m^2}} = \pm 1$$

$\Rightarrow$ $-6cm + c^2 = 9$ and $2cm + c^2 = 1$

$\Rightarrow$ $cm = -1$ and $c^2 = 3 \Rightarrow c = \pm\sqrt{3}$ and $m = \mp\dfrac{1}{\sqrt{3}}$

$\Rightarrow$ Equation of the common tangents are

$$y = -\frac{1}{\sqrt{3}}x + \sqrt{3},\ y = \frac{1}{\sqrt{3}}x - \sqrt{3}, x = 0$$

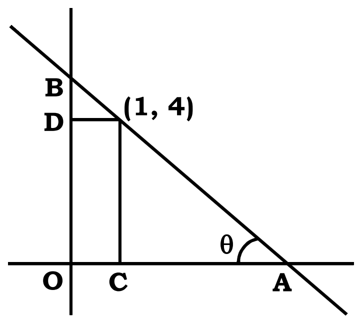
Since, the lines $y = -\dfrac{1}{\sqrt{3}}x + \sqrt{3}$ and make angle of 60^0 with $x = 0$.
The Δ PQR formed by these tangents is equilateral so that the centroid, circumcentre and orthocentre of the triangle coincide with its incentre $(1, 0)$, the centre of the circle of smaller radius inscribed in the Δ PQR.

8.

$\tan\alpha \tan\beta = -1$

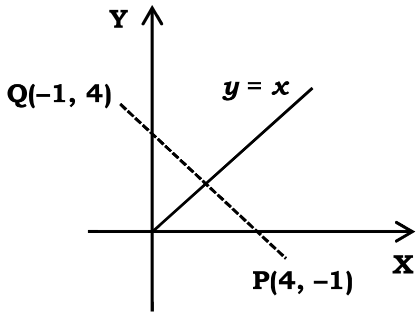
SECTION II - MATRIX MATCH TYPE

1. **A-S ; B-P ; C-Q ; D-R**
   a. OA = $1 + 4\cot\theta$, OB = $4 + \tan\theta$



OA + OB = $5 + 4\cot\theta + \tan\theta \geq 5$
$+ 2\sqrt{4\cot\theta\tan\theta}$
$= 5 + (2 \times 2) = 9$

b.



The reflection of P(4, –1) on $y = x$ is Q(–1, 4). Hence,

$PQ = \sqrt{(4+1)^2 + (-1-4)^2} = \sqrt{50} = 5\sqrt{2}$

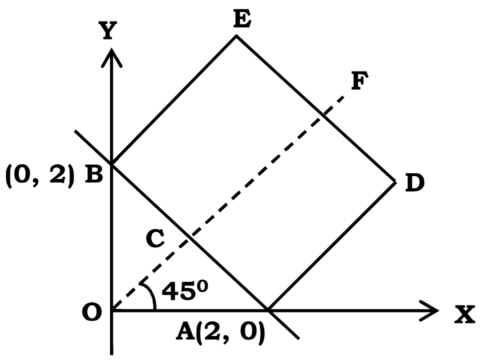
c. $AB = 2\sqrt{2}$

$OC = \sqrt{2}$

The maximum value of d is

$OF = \sqrt{2} + 2\sqrt{2}$

$= 3\sqrt{2}$



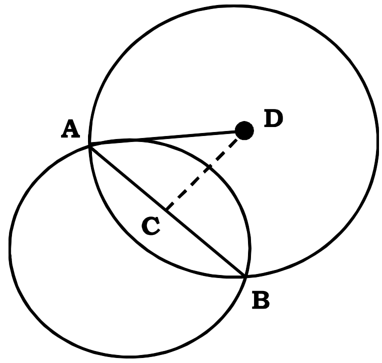
d. The given line is

$x = 4 + \frac{1}{\sqrt{2}}\left(\frac{y+1}{\sqrt{2}}\right)$ or $y = 2x - 9$

Hence, the intercept made by the x-axis is 9/2.

2. **A-S ; B-P,R,S ; C-T ; D-P,Q**

a. $CD = \sqrt{5}$ and $AC = 2$



$AD = \sqrt{5+4} = 3$

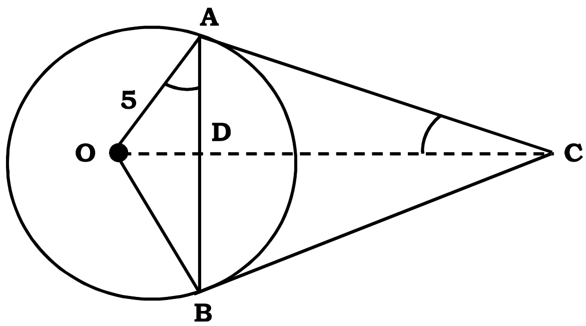
b. Distance from the centre (0, 10) of the line $y = mx$

$= \frac{10}{\sqrt{1+m^2}} \geq \text{radius} \Rightarrow |m| \leq 3$

c. $OD = 4 \Rightarrow \sin\theta = \frac{4}{5}$

So, $AC = OA \cot\theta$

$= 5 \times \frac{3}{4} = \frac{15}{4}$



∴ Area of quadrilateral

$OACB = OA \times AC = \frac{75}{4}$

d. The circle C is $x^2 + y^2 - 2x - 2y + 1 = 0$ and C_1 is $x^2 + y^2 - 2rx - 2ry + r^2 = 0$. They intersect orthogonally, if

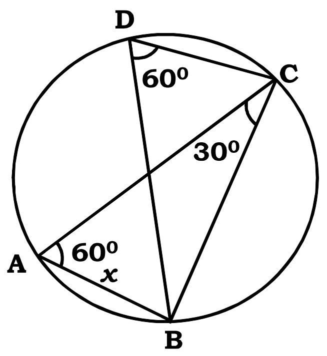
$2(r + r) = 1 + r^2$

$r^2 - 4r + 1 = 0$

So, $r = 2 \pm \sqrt{3}$

SECTION III - INTEGER TYPE

1. **1**



$\angle A = 60^0 = \angle D$

AC = 2 (Given)

$\angle ABC = 90^0$

or $x = 1$

2. **4**

(4) Let r be the radius of the required circle. Now, if two circles touch each other, then Distance between their centers = $|r \pm 2| = 5$ (Given)

$\therefore r = 3, 7$

3. **1**

If the given lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

or $\begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$

(Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

or $a(b-1)(c-1) - (c-1)$

$(1-a) - (b-1)(1-a) = 0$

or $\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$

[Dividing by $(1-a)(1-b)(1-c)$]

Adding 1 on both sides, we get

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

4. **6**

The centre of the circle in reference is O (0, 0) and given points are A (3, 4) and B (–4, 3).

We observe that

(slope of OA) × (slope of OB) = $\left(\frac{4-0}{3-0}\right) \times \left(\frac{3-0}{-4-0}\right)$

$= -1$

$\Rightarrow OA \perp OB \Rightarrow AOB = \frac{\pi}{2}$

$\Rightarrow$ The arc AB subtends an angle of $\frac{\pi}{2}$ at the centre of the circle.

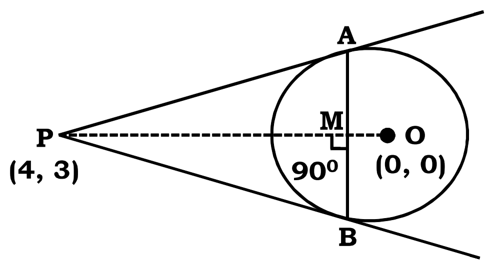
$\Rightarrow$ The chord AB subtends an angle $\frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$ at any point in the major arc.

Hence, we must have

$\angle ACB = \frac{\pi}{4} = \frac{3\pi}{2m} \Rightarrow m = 6.$

5. **3**

(3) : The equation of the chord of contact of the two tangents that can be drawn from P (4, 3) to the circle $x^2 + y^2 = 9$ is ($S_1 = 0$)



$4x + 3y - 9 = 0$ ... (1)

For area of ΔPAB, we need $|AB|$ and $|MP|$.

Here, $|OP| = \sqrt{(4-0)^2 + (3-0)^2} = 5$.

Also, $|OM|$ = distance of O (0, 0) from line (1)

$= \frac{|0+0-9|}{\sqrt{4^2+3^2}} = \frac{9}{5}$

$\therefore |AB| = 2\sqrt{(\text{radius})^2 - OM^2}$

$= 2\sqrt{3^2 - \left(\frac{9}{5}\right)^2} = 2\sqrt{\frac{144}{25}} = \frac{24}{5}$

and $|MP| = |OP| - |OM|$

$$= 5 - \frac{9}{5} = \frac{16}{5}.$$

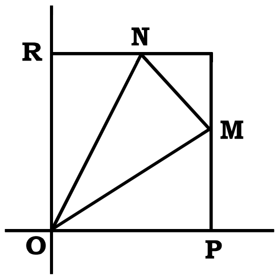
Hence area $\Delta PAB = \frac{1}{2}|AB||MP|$

$$= \frac{1}{2}\left(\frac{24}{5}\right)\left(\frac{16}{5}\right) = \frac{192}{25} \text{ square units}$$

$$\Rightarrow \frac{64\Delta}{25} = \frac{192}{25} \Rightarrow \Delta = 3.$$

6. **4**

We can assume that, OP and OR are x-axis and y-axis respectively.



Let OP = a, then area (square OPQR) = a^2

Coordinate of M and N are $\left(a, \frac{a}{2}\right)$ and $\left(\frac{a}{2}, a\right)$ respectively.

$$\therefore \text{Area } (\Delta OMN) = \frac{1}{2}\begin{vmatrix} a & a/2 \\ a/2 & a \end{vmatrix} = \frac{3a^2}{8}$$

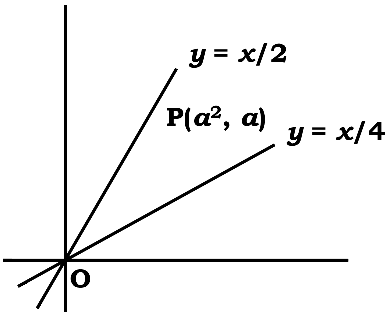
$$\therefore \frac{8}{3} = \frac{\lambda}{6}$$

$\therefore \lambda = 16$

$$\Rightarrow \frac{\lambda}{4} = \frac{16}{4} = 4$$

7. **6**

We have, $a - \frac{a^2}{4} > 0$ and $a - \frac{a^2}{2} < 0$



8. **6**

The circles are given as $x^2 + y^2 = 12$ and $x^2 + y^2 - 5x + 4y - 2 = 0$ Common chord AB is $5x - 3y - 10 = 0$, let the coordinate of P be (α, β). Equation of the chord of contact of $f(\alpha, \beta)$ with respect to $x^2 + y^2 = 12$ is $x\alpha + y\beta - 12 = 0$.

On comparing the coefficients of common chord AB and chord of contact is $\frac{\alpha}{5} = \frac{\beta}{-3} = \frac{-12}{-10}$, we get

$\alpha = 6$

$\therefore$ x–coordinate is 6.