PART C - MATHS

61. **b)** **10**

The numbers greater than 300000, have '3' fixed in extreme left place

3 X X X X X

The remaining 5 letters can be arranged in places marked 'X' in $\frac{\lfloor 5}{3\lfloor 2}$ ways.

($\therefore$ 1 is repeated thrice and 2 is repeated twice.)

62. **b)** $\mathbf{12^6 - P(12, 6)}$

When repetition is allowed, the number of words formed is 12^6 and when there is no repetition, the number of words formed is P(12, 6)

63. **d)** $\dfrac{\lfloor a+2b+3c+d}{\lfloor a\,(\lfloor b)^2\,(\lfloor c)^3}$

We have $a + 2b + 3c + d$ objects to be arranged.

Out of these one object is repeated 'a' times, two objects are repeated b times each, three objects are repeated c times each.

$\therefore$ Required number of arrangements

$$= \frac{\lfloor a+2b+3c+d}{\lfloor a\,\lfloor b\,\lfloor b\,\lfloor c\,\lfloor c\,\lfloor c} = \frac{\lfloor a+2b+3c+d}{\lfloor a\,(\lfloor b)^2\,(\lfloor c)^3}$$

64. **c)** $\dfrac{12!}{5!3!2!} \times \dfrac{^{13}P_3}{3!}$

12 letters (other than C, C, C) can be arranged in a row in $\frac{12!}{5!3!2!}$ ways and in any such arrangement, three C's can be placed in 13 gaps available (including extremes) in

$^{13}C_3 = {^{13}P_3}/\lfloor 3$ ways.

65. **c)** **zero**

Since $x^2 - y^2$ is even, therefore, either bond a and y are even or both x and y are odd. In either case both $x + y$ and $x - y$ are even.

So, 2 divides $x + y$ and also 2 divides $x - y$

$\Rightarrow$ 2×2 divides $(x + y)(x - y)$

$\Rightarrow$ 4 divides $x^2 - y^2$.

But 4 does not divide 3906. Hence, the gives equation has no solution of the said type

66. **b)** $\dfrac{\lfloor 2n}{\lfloor n\,2^n}$

Required number

$$= \frac{^{2n}C_2 \times {^{2n-2}C_2} \times {^{2n-4}C_2} \times \ldots \times {^2C_2}}{\lfloor n}$$

$$= \frac{\lfloor 2n}{(\lfloor 2)^n\,\lfloor n} = \frac{\lfloor 2}{2^n\lfloor n}$$

(Saince each group contains two persons, the groups are interchangeable, therefore, we have divide by $\lfloor n$)

67. **b)** $\mathbf{^{m+n+k}C_3 - {^mC_3} - {^nC_3} - {^kC_3}}$

Total number of points is $m + n + k$. If all the points were such that no three of them are in the same line, they would have given $^{m+n+k}C_3$ lines. m points on l_1 give us no triangles ; n points on l_2 also do not give us any triangle. Similarly, k points on l_3 make no triangle. So, the maximum number of triangles that can be formed = $^{m+n+k}C_3 - {^mC_3} - {^nC_3} - {^kC_3}$.

68. **b)** **18**

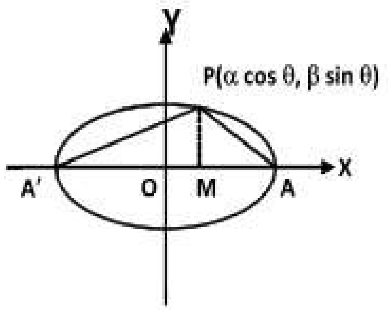
Each selection of a pair of line from one set and a pair of lines from the second set gives us a parallelogram.

$\therefore$ Required number of parallelogram

$= {^4C_2} \times {^4C_2} = 6 \times 3 = 18$

69. **a)** ***ab***

If P is the point (a cos θ, b sin θ) then area



$\Delta APA' = \frac{1}{2}|AA'|(b \sin \theta)$

$= \frac{1}{2}(2a)(b \sin \theta) = ab \sin \theta$

which is maximum when sinθ = 1.

70. **c)** $\boldsymbol{\frac{x^2}{40} + \frac{y^2}{10} = 1}$

The major axis of the ellipse is along the line $x - y = 0$ and the minor axis is along the line $x + y - 2 = 0$. The centre of the ellipses is the point of intersection of these two lines, $x - y = 0$ and $x + y - 2 = 0$, which is (1, 1)

71. **a)** $\boldsymbol{\frac{1}{2x^2} + \frac{1}{4y^2} = 1}$

Any point on the given ellipse $\frac{x^2}{2} + \frac{y^2}{1} = 1$ is $(\sqrt{2} \cos \theta, \sin \theta)$.

Equation of the tangent to the ellipse at this point is

$\frac{x}{\sqrt{2} \sec \theta} + \frac{y}{\operatorname{cosec} \theta} = 1$ or

$\frac{x}{\sqrt{2} \cos \theta} + \frac{y}{\operatorname{cosec} \theta} = 1$

This tangent meets X-axis in the point P($\sqrt{2}$ sec θ, 0) and Y-axis in the point Q(0, cosec θ).

Let (α, β) be the mid-point o [PQ], then

$\alpha = \frac{\sqrt{2}\cos\theta + 0}{2}$ and $\beta = \frac{0 + \operatorname{cosec}\theta}{2}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}\alpha}$ and $\sin\theta = \frac{1}{2\beta}$

$\Rightarrow \left(\frac{1}{\sqrt{2}\alpha}\right)^2 + \left(\frac{1}{\sqrt{2}\beta}\right)^2 = 1$

($\because \cos^2\theta + \sin^2\theta = 1$)

Hence locus of (α, β) is $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$

72. **c)** **17**

We know that $(\underline{|n})^2$ is divisible by $10^2 = 100$ for $n \geq 5$.
So, we need compute

$(\underline{|1})^2 + (\underline{|2})^2 + (\underline{|3})^2 + (\underline{|4})^2$
$= 1^2 + 2^2 + 6^2 + 24^2$
$= 1 + 4 + 36 + 576 = 617$

Hence, required remained = 17

73. **a)** **70**

Since 38808 = 8 × 4851
= 8 × 9 × 539 = 8 × 9 × 7 × 7 × 11
$= 2^3 \times 3^2 \times 7^2 \times 11^1$
So, number of divisors
= (3 + 1)(2 + 1) (2 + 1) (1 + 1) = 72
This includes two divisors 1 and 38808.
Hence, the required number of divisors
= 72 – 2 = 70

74. **c)** **8 metres**

For the ellipse in reference, $2a = 400$ and $b = 10$ and hence, the equation of the ellipse can be written as $\frac{x^2}{(200)^2} + \frac{y^2}{10^2} = 1$. The point which is 80 m from one end is (200 – 80)m = 120 m from the centre. Substituting $x = 120$ in the equation, we get

$\left(\frac{120}{200}\right)^2 + \frac{y^2}{10^2} = 1 \quad \Rightarrow y = 8.$

75. **c)** $\boldsymbol{\frac{1}{\sqrt{2}}}$

The foci are $F_1(ae, 0)$ and $F_2(-ae, 0)$ and

an end of the minor axis is B(0, b). Since $\angle F_1BF_2 = 90^0$, therefore,
(slope of BF1) × (slope of BF_2) = –1

$\Rightarrow \quad \frac{b}{-ae} \times \frac{b}{ae} = -1 \quad \Rightarrow b^2 = a^2e^2$

$\Rightarrow \quad a^2(1 - e^2) = a^2e^2 \quad \Rightarrow 2e^2 = 1$

$\Rightarrow \quad e = \frac{1}{\sqrt{2}}$

76. **d)** $-\left(\frac{a^2 + b^2}{b}\right)$

Equations of normals at the points θ, φ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are respectively. $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ ...(i)

and $\frac{ax}{\sec\phi} + \frac{by}{\tan\phi} = a^2 + b^2$ ...(ii)

(i) and (ii) can be written as
$ax + by \operatorname{cosec}\theta = (a^2 + b^2)\sec\theta$ ...(iii)
$ax + by \operatorname{cosec}\phi = (a^2 + b^2)\sec\phi$ ...(iv)
For the point of intersection, subtracting ...(iv)
$b(\operatorname{cosec}\theta - \operatorname{cosec}\phi) = (a^2 + b^2)$ $(\sec\theta - \sec\phi)$

$\Rightarrow \quad by\left(\operatorname{cosec}\theta - \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right)\right)$

$= (a^2 + b^2)\left(\sec\theta - \sec\left(\frac{\pi}{2} - \theta\right)\right)$

$\Rightarrow \quad by(\operatorname{cosec}\theta - \sec\theta)$
$= (a^2 + b^2)(\sec\theta - \operatorname{cosec}\theta)$

$\Rightarrow \quad y = -\left(\frac{a^2 + b^2}{b}\right) \quad \Rightarrow k = -\left(\frac{a^2 + b^2}{b}\right)$

77. **a)** $\frac{\pi}{6}$

Equation of the tangent at the point $(3\sqrt{3}\cos\theta, \sin\theta)$ is

$\frac{x(3\sqrt{3}\cos\theta)}{27} + \frac{y\sin\theta}{1} = 1$ or

$\frac{x}{3\sqrt{3}\sec\theta} + \frac{y\sin\theta}{\operatorname{cosec}\theta} = 1$

Let S = sum of intercepts of line (i) on the coordinate axes, then S = $3\sqrt{3}$ $\sec\theta + \operatorname{cosec}\theta$ for least S, $\frac{ds}{d\theta} = 0$

Which is quadratic giving two values of m which are the slopes of the two tangents. Say, these are m_1, m_2 then
$m_1 + m_2 = 4$, $m_1 m_2 = 1$
If θ is the (acute) angle between the tangents, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \frac{\sqrt{(m_1 - m_2)^2 - 4m_1m_2}}{|1 + m_1m_2|}$$

$$= \frac{\sqrt{16 - 4}}{1 + 1} = \frac{2\sqrt{3}}{2} = \sqrt{3}.$$

78. **a)** $y - \frac{1}{5} = -\frac{3}{4}\left(x - \frac{1}{2}\right)$

79. **b)** **(1, 1)**

80. **a)** $3x - 4y = 4$

By T = S_1, the equation of chord whose mid point is (α, β) is $3x\alpha - 2y\beta + 2(x + \alpha) - 2(y + \beta) = 0$

$\Rightarrow \quad x(3\alpha + 2) - y(2\beta + 3) = 2$ as it is parallel to $y = 2x$

$\therefore \quad 3\alpha - 4\beta = 4$

$\therefore$ Required locus is $3x - 4y = 4$

81. **c)** **1106**

Number of digits in the number	Digit in units place	Number of numbers
1	5	1
2	0	${}^9P_1 = 9$
2	5	${}^8P_1 = 8$
3	0	${}^9P_2 = 72$
3	5	8 × 8 = 64
4	0	${}^9P_3 = 504$
4	5	8 × 8 × 7 = 448
Total		1106

82. **b)** **6**

Since $4n + 2 = 2 \times (2n + 1)$, therefore, we have to consider only those divisors of 2880 which are odd multiples of 2.

($\because$ $2n + 1$ is an odd number)

Now $2880 = 2^6 \times 3^2 \times 5^1$

Hence, any divisor of the said type is of the form $2 \times 3^a \times 5^b$ where $a\ \varepsilon \{0, 1, 2\}$ and $b\ \varepsilon \{0, 1\}$.

Thus, a has 3 choices and b has 2 choices.

$\therefore$ Required number of divisors = 3 × 2 = 6.

83. **d)** $2\left(\underline{|n+1}\right)^2$

First of all, we note that there are two exclusive ways of arranging the balls.

(i) White balls may occupy odd places and black balls may occupy even places.

(i) white balls may occupy even places and black balls may occupy add places.

In either case, the number of arrangements

$= {}^{n+1}P_{n+1} \times {}^{n+1}P_{n+1} = \underline{|n+1}\,\underline{|n+1}$

$\therefore$ Required number of ways

$= 2\ \underline{|n+1}\,\underline{|n+1} = 2\left(\underline{|n+1}\right)^2$.

84. **b)** $^{14}C_3 - 4 \times {}^8C_3$

Required number of ways

= Coeff. of x^{15} in $(x^1 + x^2 + x^3 + x^4 + x^5 + x^6)^4$

= Coeff. of x^{15} in $x^4(1 + x + x^2 + \ldots + x^5)^4$

= Coeff. of x^{11} in $(1 + x + x^2 + x^3 + x^4 + x^5)^4$

= Coeff. of x^{11} in $(1 - x^6)^4 (1 - x)^{-4}$

= Coeff. of x^{11} in $(1 - 4x^6 + 6x^{12} - \ldots)(1 - x)^{-4}$ = Coeff.of x^{11} in $(1 - x)^{-4}$

$- 4 \times$ Coeff. of x^5 in $(1 - x)^{-4}$

$= {}^{11+4-1}C_{11} - 4 \times {}^{5+4-1}C_5$

$= {}^{14}C_{11} - 4 \times {}^8C_5$

$= {}^{14}C_3 - 4 \times {}^8C_3$

85. **a)** $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$

The given ellipse is $3x^2 + 4y^2 = 12$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Which is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a^2 = 4,\ b^2 = 3$

$\Rightarrow$ $a = 2,\ b = \sqrt{3}$

Its eccentricity 'e' is given by

$b^2 = a^2(1 - e^2)$ or $3 = 4(1 - e^2)$

$\Rightarrow$ $e = \frac{1}{2}$

$\therefore$ Foci of the given ellipse are at $(\pm ae, 0) = (\pm 1, 0)$.

Since the hyperbola is confocal with the given ellipse, therefore, for the hyperbola $1 = (\sin\theta)e'$ ...(i)

($\because$ length of transverse axis = $2\sin\theta$)

Where e' is the eccentricity of the hyperbola. If b' is the length of semi-conjugate axis, then

$b'^2 = \sin^2\theta(e'^2 - 1)$

$\Rightarrow$ $b'^2 = e'^2\sin^2\theta - \sin^2\theta$

$= 1 - \sin^2\theta$

$= \cos^2\theta$ (using (i))

Hence, the equation of hyperbola in reference is $\frac{x^2}{\sin^2\theta} - \frac{x^2}{\cos^2\theta} = 1$

86. **b)** $\sqrt{\frac{3}{2}}$

Given curve is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$;

differentiating $w.r.t.x$, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{b^2}{a^2}\left(\frac{x}{y}\right)$$

$\Rightarrow$ Slope of the tangent at P(6, 3)

$$= \frac{b^2}{a^2}\left(\frac{6}{3}\right) = \frac{2b^2}{a^2}$$

$\Rightarrow$ Slope of the normal at P = $\frac{-a^2}{2b^2}$.

Hence equation of the normal at P is

$$y - 3 = \frac{-a^2}{2b^2}(x - 6).$$

But (9, 0) lies on this normal, therefore,

$$0 - 3 = \frac{-a^2}{2b^2}(9 - 6)$$

$\Rightarrow \quad 2b^2 = a^2$

$\Rightarrow \quad 2a^2(e^2 - 1) = a^2 \qquad \Rightarrow 2e^2 = 3$

$e = \sqrt{\frac{3}{2}}$

87. **b)** $\mathbf{\frac{2\sqrt{6}}{7}}$

The ellipse contains the points (7, 0) and (0, –5). If the equation of the ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\frac{7^2}{a^2} = 1$ and $\frac{(-5)^2}{b^2} = 1$

$\Rightarrow \quad a^2 = 49$ and $b^2 = 25 \ \Rightarrow a^2(1 - e^2) = 25$

$\Rightarrow \quad 49(1 - e^2) = 25.$

88. **d)** $\mathbf{\left(4, -\sqrt{6}\right)}$

89. **a)** $\mathbf{\frac{x^2}{4} - \frac{y^2}{5} = 1}$

Required equation of locus is

$\left|\sqrt{(x-3)^2 + y^2} - \sqrt{(x+3)^2 + y^2}\right| = 4$

$\Leftrightarrow \quad \sqrt{(x-3)^2 + y^2} = \sqrt{(x+3)^2 + y^2} \pm 4$

$\Leftrightarrow \quad (x - 3)^2 + y^2 = 16 + (x + 3)^2 + y^2$

$\pm 8\sqrt{(x+3)^2 + y^2}$

$\Leftrightarrow \quad -12x - 16 = \pm 8\sqrt{(x+3)^2 + y^2}$

$\Leftrightarrow \quad (3x + 4)^2 = 4(x + 3)^2 + 4y^2$

$\Leftrightarrow \quad 5x^2 - 4y^2 = 20.$

90. **a)** $\mathbf{\left(0, \sqrt{7}\right), \left(0, -\sqrt{7}\right)}$

For the conic $\frac{x^2}{9} + \frac{y^2}{16} = 1$,

$b = 3,\ a = 4$ …(i)

and $ae = \sqrt{7}$

The second conic is $\frac{x^2}{B^2} + \frac{y^2}{A^2} = 1$

Where $A^2 = 16 + t^2$ and $B^2 = 9 + t^2$

$\Rightarrow \quad A^2 - B^2 = 16 - 9 = 7$. Also $B^2 = A^2(1 - e'2)$

Where e' is the eccentricity of the second conic

$\Rightarrow \quad A^2e'^2 = A^2 - B^2$

$Ae' = (16 - 9)^{1/2} = \sqrt{7}$