PART C - MATHS

SECTION I - MULTIPLE ANSWER CORRECT TYPE

1. **a)** $\boldsymbol{a = 2}$

c) $\mathbf{L = \frac{1}{64}}$

$$L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 0} \frac{1}{x^2\left(a + \sqrt{a^2 - x^2}\right)} - \frac{1}{4x^2}$$

$$= \lim_{x\to 0} \frac{(4-a) - \sqrt{a^2 - x^2}}{4x^2\left(a + \sqrt{a^2 - x^2}\right)}$$

Numerator $\to$ 0 if $a = 2$ and then L = $\frac{1}{64}$.

2. **a)** $\boldsymbol{\lim_{x\to 0} f(x)}$ **exists for** $\boldsymbol{n > 1}$

For $n > 1$

$\lim_{x\to 0} x^n \sin(1/x^2) = 0 \times$ (any value between –1 to 1) = 0

For $n < 0$,

$\lim_{x\to 0} x^n \sin(1/x^2) = \infty \times$ (any value between –1 to 1) = ∞

3. **b) g is differentiable while *f* is not**

d) g is differentiable and g′ is continuous

We have $g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

If $x \neq 0$, $g'(x) = x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right)$

$= -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$

which exists for $\forall\ x \neq 0$

If $x = 0$,

then $g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0}$

$= \lim_{x\to 0} \frac{x^2 \sin(1/x) - 0}{x - 0}$

$= \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$

$\Rightarrow g'(x) = \begin{cases} -\cos\left(\frac{1}{x}\right) + 2x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

At $x = 0$, $\cos\left(\frac{1}{x}\right)$ is not continuous, therefore $g'(x)$ is not continuous at $x = 0$.

At $x = 0$,

$Lf' = \lim_{x\to 0} \frac{0 - (-x)\sin \sin\left(-\frac{1}{x}\right)}{x} = \sin\left(\frac{1}{x}\right)$

which does not exist.

4. **c)** $\mathbf{R - \{-1\}}$

The given function is

$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \le 1 \\ \frac{1}{2}(|x| - 1), & \text{if } |x| > 1 \end{cases}$

$\Rightarrow f(x) = \begin{cases} \frac{1}{2}(-x-1), & \text{if } x < -1 \\ \tan^{-1} x & \text{if } -1 \le x \le 1 \\ \frac{1}{2}(x-1), & \text{if } x > 1 \end{cases}$

Clearly, $f(x)$ is discontinuous at $x = 1$ and -1 and hence non-differentiable at $x = 1$ and -1. Hence, $f(x)$ is differentiable for $R - \{-1, 1\}$.

5. **a)** $\boldsymbol{f(4) = f(-4)}$

c) $\boldsymbol{f'(4) + f'(4) = 0}$

$f(x - y)$, $f(x) f(y)$ and $f(x + y)$ are in A.P.

$\Rightarrow f(x + y) + f(x - y) = 2f(x)f(y)$ for all x, y

Putting $x = 0$, $y = 0$ in (1)

we get $f(0) + f(0) = 2f(0)f(0)$

$\Rightarrow f(0) = 1 \quad (\because f(0) \neq 0)$

Putting $x = 0$, $y = x$,

we get $f(x) + f(-x) = 2f(0)f(x)$

$\Rightarrow f(x) = f(-x)$

$\Rightarrow f(4) = f(-4)$, $f(3) = f(-3)$

Differentiating (1) w.r.t. x, $f'(x) + f'(-x) = 0$

$\Rightarrow f'(4) + f'(-4) = 0$ and $f'(2) = f'(-2)$

6. **a) (2, 8/3)**
b) (3, 7/2)
Since the intercepts are equal in magnitude but opposite in sign

$\Rightarrow \left.\frac{dy}{dx}\right|_P = 1$

now $\frac{dy}{dx}$ $x^2 - 5x + 7 = 1$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x = 2$ or 3.

7. **a) $x = 1$**
b) $x = 2$
c) $x = 3$

$f(x) = \frac{2-x}{\pi}\cos\pi(x+3) + \frac{1}{\pi^2}\sin\pi(x+3)$

$f'(x) = -\frac{1}{\pi}\cos\pi(x+3) - (2-x)\sin\pi(x+3)$

$+ \frac{1}{\pi}\cos\pi(x+3)$

$= (x-2)\sin\pi(x+3) = 0$

$x = 2, 1, 3$

$f''(x) = \sin\pi(x+3) + \pi(x-2)\cos\pi(x+3)$

$f''(1) = -\pi < 0$, $f''(2) = 0$, $f''(3) = \pi > 0$

$\therefore$ $x = 1$ is a maximum and $x = 3$ is a minimum, hence $x = 2$ is the point of inflection.

8. **a) $(-\infty, -4/3)$**
d) $(2, \infty)$
We have $f(x) = (4a-3)(x + \log 5) + 2(a-7)$ $\cot\frac{x}{2}\sin^2\frac{x}{2}$

$= (4a-3)(x + \log 5) + (a-7)\sin x$

$\Rightarrow f'(x) = (4a-3) + (a-7)\cos x$

If $f(x)$ does not have critical points, then $f'(x) = 0$ does not have any solution in R.

Now, $f'(x) = 0 \Rightarrow \cos x = \frac{4a-3}{7-a}$

$\Rightarrow \left|\frac{4a-3}{7-a}\right| \le 1$ $\quad$ $[\because |\cos x| \le 1]$

$\Rightarrow -1 \le \frac{4a-3}{7-a} \le 1 \Rightarrow a - 7 \le 4a - 3 \le 7 - a$

$\Rightarrow a \ge -4/3$ and $a \le 2$

Thus, $f'(x) = 0$ has solutions in R if $-4/3 \le a \le 2$

So, $f'(x) = 0$ is not solvable in R if $a < -4/3$ or $a > 2$, i.e.,

$a \in (-\infty, -4/3) \cup (2, \infty)$.

9. **a) A = –2/3**
d) B = 8/3 e^3

$\frac{2x}{(x-1)(x-4)} = \frac{C}{x-1} + \frac{D}{x-4}$

$2x = C(x-4) + D(x-1)$

$\therefore C = -2/3, D = 8/3$

$\therefore \int \frac{e^{x-1}}{(x-1)(x-4)} 2x\,dx$

$= \int e^{x-1}\left(\frac{-2/3}{x-1} + \frac{8/3}{x-4}\right)dx$

$= -\frac{2}{3}F(x-1) + \frac{8}{3}e^3 F(x-4) + C$

$\therefore A = -2/3, B = 8/3\, e^3$

10. **c) A = 2^{100} B**

Given $A = \int_0^1 x^{50}(2-x)^{50}\,dx$;

$B = \int_0^1 x^{50}(1-x)^{50}\,dx$

In A, put $x = 2t \Rightarrow x = 2dt$

$\Rightarrow A = 2\int_0^{1/2} 2^{50}.\, t^{50}\, 2^{50}(1-t)^{50}\,dt$

Now $B = 2\int_0^{1/2} x^{50}(1-x)^{50}\,dx$

$\left[\text{using} \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x)\right]$

From equations (1) and (2), we get

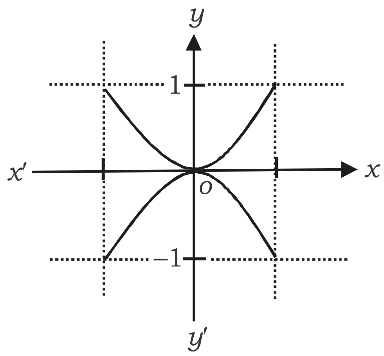
$A = 2^{100}B$

SECTION II - MATRIX MATCH TYPE

1. **A-P,Q,R ; B - P,S ; C-R,S ; D-P,Q**

A) $x|x|$

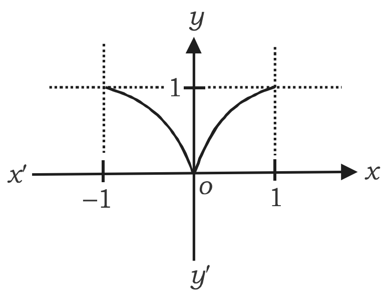
p, q, r. $y = x|x|$



From the graph, $f(x)$ is continuous and differentiable in (–1, 1). Also $f(x)$ is strictly increasing.

B) $\sqrt{|x|}$

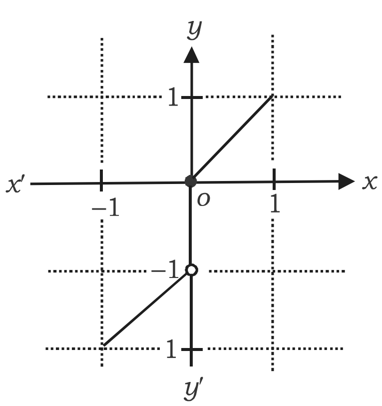
p, s. $y = \sqrt{|x|}$



From the graph, $f(x)$ is continuous in (– 1, 1), but non-differentiable at $x = 0$.

C) $x + [x]$

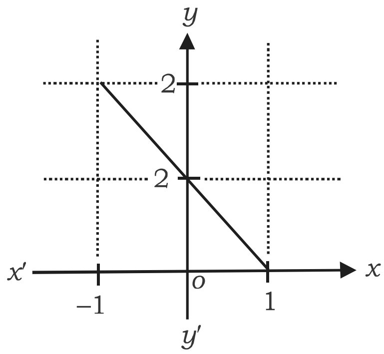
r, s. $y = x + [x]$



From the graph, $f(x)$ is discontinuous at $x = 0$. Also $f(x)$ is increasing.

D) $[x - 1]$

p, q. $y = |x - 1|$



From the graph, $f(x)$ is continuous and differentiable in (– 1, 1).

2. ***A.* → R, B. → S, C → Q, D → P.**

A) $\int \frac{e^{2x}-1}{e^{2x}+1}dx$

$= \int \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$

$= \int \frac{(e^{x}+e^{-x})}{e^{x}+e^{-x}}dx$

$= \log(e^{x} + e^{-x})$

$= \log(e^{2x} + 1) - x + C$

B) $\int \frac{1}{(e^{x}+e^{-x})^2}dx$

$I = \int \frac{1}{(e^{x}+e^{-x})^2}dx = \int \frac{e^{2x}}{(e^{2x}+1)^2}dx$

Put $e^{2x} + 1 = t \Rightarrow 2e^{2x}\,dx = dt$, we get

$\Rightarrow I = \frac{1}{2}\int \frac{1}{t^2}dt = -\frac{1}{2}\frac{1}{t} + C = -\frac{1}{2(e^{2x}+1)} + C.$

C) $\int \frac{e^{-x}}{1+e^{x}}dx$

$I = \int \frac{e^{-x}}{1+e^{x}}dx = \int \frac{e^{-x}e^{-x}}{e^{-x}+1}dx$

Put $e^{-x} + 1 = t \Rightarrow -e^{-x}\,dx = dt$

$\Rightarrow I = \int \frac{(t-1)}{t}dt = \int \left(\frac{1}{t}-1\right)dt$

$= \log t - t + C$

$= \log(e^{-x} + 1) - (e^{-x} + 1) + C$

$= \log(e^{x} + 1) - x - e^{-x} - 1 + C$

$= \log(e^x + 1) - x - e^{-x} + C$

D) $\int \frac{1}{1+e^{2x}} dx$

$I = \int \frac{1}{\sqrt{1-e^{2x}}} dx = \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}} dx$

Put $e^{-x} = t \Rightarrow -e^{-x} dx = dt$,

$\Rightarrow I = -\int \frac{1}{\sqrt{t^2-1}} dt$

$= -\log\left[t + \sqrt{t^2-1}\right] + C$

$= -\log\left[e^{-x} + \sqrt{e^{-2x}-1}\right] + C$

$= \log\left[\frac{1}{e^x} + \frac{\sqrt{1-e^{2x}}}{e^x}\right] + C$

$= -\log\left[1 + \sqrt{1-e^{2x}}\right] + \log e^x + C$

$= x - \log\left[1 + \sqrt{1-e^{2x}}\right] + C$

SECTION III - INTEGER TYPE

1. **4**

$\lim_{x\to 0} \frac{e^{2x} - (1+4x)^{1/2}}{\ln(1-x^2)} = \lim_{x\to 0} \frac{e^{2x} - (1+4x)^{1/2}}{\frac{\ln(1-x^2)}{-x^2}(-x^2)}$

$= \lim_{x\to 0} \frac{(1+4x)^{1/2} - e^{2x}}{x^2}$

$= \lim_{x\to 0} \frac{\left(1 + \frac{1}{2}4x + \frac{1}{2}\left(\frac{1}{2}-1\right)\frac{1}{2!}16x^2 + \ldots\right) - \left(1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \ldots.\right)}{x^2}$

$= -2 - 2 = -4$

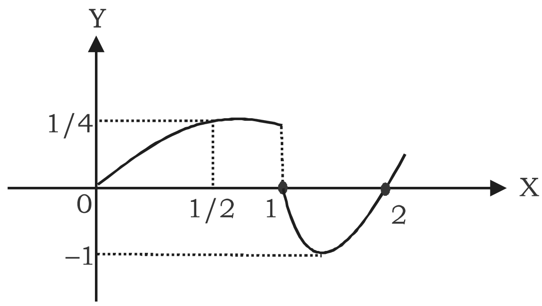
Thus, P = 4.

2. **1**

$f'(x) = 1 - 2x > 0$ If $x < \frac{1}{2}$ and < 0 if $x > \frac{1}{2}$

$\therefore f(x)$ is increasing in $0 \le x \le \frac{1}{2}$

$\Rightarrow \max f(t) = f(x)$



Also maximum of

$f(x) = f\left(\frac{1}{2}\right) = \frac{1}{4}$ $(0 \le x \le 1)$

$\therefore \text{Max } f(t) = f\left(\frac{1}{2}\right) = \frac{1}{4}$ If $\frac{1}{2} \le x \le 1$

so $g(x) = \begin{cases} x - x^2, & 0 \le x \le \frac{1}{2} \\ \frac{1}{4}, & \frac{1}{2} \le x \le 1 \\ \sin \pi x, & x > 1 \end{cases}$

It is clear from the graph that g(x) is continuous every where except at $x = 1$, hence not differentiable at $x = 1$

Thus, P = 1.

3. **1**

We have $f(x) = -1, -2 \le x \le 0$

$= x - 1, 0 \quad 0 < x \le 2$

and $g(x) = f(|x| + |f(x)|$

Hence g(x) involves $|x|$ and $|x - 1|$ or $|-1| = 1$

Therefore we should divide the given interval (– 2, 2) into the following intervals.

I_1	I_2	I_3
[–2, 2] = [–2, 0)	[0, 1)	[1, 2]
x = –ve	+ ve	+ ve
$\|x\| = -x$	x	x
$f(x) = -1$	$x - 1$	$x - 1$
$f(\|x\|) = -1$	$= x - 1$	$= x - 1$
$\|f(x)\| = \|-1\|$	$\|x - 1\|$	$\|x - 1\|$
$= 1$	$= -(x - 1)$	$= x - 1$

$\therefore$ Using above we get

$g(x) = f|x| + |f(x)|$

$= -1 + 1 = 0$ in I_1

$= x - 1 - (x - 1) = 0$ in I_2

$= x - 1 + x - 1 = 2(x - 1)$ in I_3

Hence g(x) is defined as follows :

$$g(x) = \begin{cases} 0, & -2 \le x < 1 \\ 2(x-1), & 1 \le x \le 2 \end{cases}$$

Lg′ (1) = 0; Rg′ (1) = 2 (not equal)

Hence g(x) is not differentiable at $x = 1$.

Thus A = 1.

4. **0**

As we know

$1 < \sqrt[3]{\frac{\pi}{2}} < 2$, $\quad \therefore$ If $x = \sqrt[3]{\frac{\pi}{2}} \Rightarrow [x] = 1$

So, $f(x) = \cos\left\{\frac{\pi}{2} - x^3\right\} = \sin x^3$

$\Rightarrow f'(x) = \cos x^3 . 3x^2$

$\therefore f'\left(\sqrt[3]{\frac{\pi}{2}}\right) = 3\left(\frac{\pi}{2}\right)^{2/3} . \cos\frac{\pi}{2} = 0 \Rightarrow f'\left(\sqrt[3]{\frac{\pi}{2}}\right) = 0$

5. **3**

$z = (\cos x)5$, $y = \sin x$

$\frac{dz}{dx} = -5\cos^4 x.\sin x$, $\frac{dy}{dx} = \cos x$

$\therefore \frac{dz}{dy} = -5\cos^2 x.\sin x$

now $\frac{d^2z}{dy^2} = \frac{d}{dx}\left(\frac{dz}{dy}\right)\frac{dx}{dy}$

$= -5\frac{d}{dx}[\cos^3 x.\sin x]\frac{1}{\cos x}$

$= -5[\cos^4 x - 3\sin^2 x.\cos^2 x]\frac{1}{\cos x}$

$= -5(\cos^3 x - 3\sin^2 x.\cos x)$

$= -5(\cos^3 x - 3\cos x(1 - \cos^2 x))$

$= -5(4\cos^3 x - 3\cos x) = -5\cos 3x$

$\therefore \left.\frac{d^2z}{dy^2}\right|_{x=\frac{2\pi}{9}} = -5\cos 120^0 = \frac{5}{2}$

Thus, P – Q = 5 – 2 = 3

6. **3**

$\int x^{51}\left(\tan^{-1}x + \cot^{-1}x\right)dx$

$= \int x^{51}.\frac{\pi}{2}dx \qquad \left\{\because \tan^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right\}$

$= \frac{\pi x^{52}}{104} + C = \frac{x^{52}}{52}(\tan^{-1}x + \cot^{-1}x) + C.$

Thus M = 52, N = 52,

P = 1, Q = 1

$\Rightarrow \frac{M}{N} + P + Q = 1 + 1 + 1 = 3$

7. **2**

Here $I = \int \sin^{-1}(\cos x)dx$

$\Rightarrow \int \sin^{-1}\sin\left(\frac{\pi}{2} - x\right)dx$

$\Rightarrow \int\left(\frac{\pi}{2} - x\right)dx = \frac{\left(\frac{\pi}{2} - x\right)}{-2} + C$

Thus K = 2

8. **7**

Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}dx$

$= \int_0^{\pi/4}\frac{\log(1+\tan\theta)}{(1+\tan^2\theta)}\sec^2\theta\, d\theta$

[Putting $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$)

$= \int_0^{\pi/4}\log(1+\tan\theta)d\theta$

$= \int_0^{\pi/4}\log\left[1+\tan\left(\frac{\pi}{4} - \theta\right)\right]d\theta$

$= \int_0^{\pi/4}\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right)d\theta = \int_0^{\pi/4}\log\left(\frac{2}{1+\tan\theta}\right)d\theta$

$= \log 2\int_0^{\pi/4}1d\theta - I \Rightarrow 2I = \frac{\pi}{4}\log 2$

$\therefore I = \frac{\pi}{8}\log 2$

Thus, A = 1, B = 8 and C = 2

$\Rightarrow$ A + B – C = 1 + 8 – 2 = 7