Math Paper II

SOLUTIONS

MULTIPLE ANSWER TYPE

1. **a) , b)**

The curve should be symmetrical about the point (2,1). Then there exists at least two parallel tangents

2. **a), b), c), d)**

$$f\left(f(x)\right) = 1 - x$$

$$f\left(f\left(f(x)\right)\right) = 1 - f(x)$$

$$f\left(1 - x\right) = 1 - f(x)$$

$$f(x) + f(1 - x) = 1$$

$$f'(x) - f'(1 - x) = 0$$

$$f''(x) + f''(1 - x) = 0$$

$$f''\left(\tfrac{1}{4}\right) + f''\left(\tfrac{3}{4}\right) = 0$$

$$J = \int_0^1 f(x)dx = \int_0^1 f(1 - x)dx$$

$$2J = \int_0^1 \left(f(x) + f(1 - x)\right)dx = \int_0^1 1\,dx$$

$$f''\left(\tfrac{1}{5}\right) = 0,$$

$f'''(x) = 0$ has atleast one root in $x \in \left(\tfrac{1}{2}, \tfrac{4}{5}\right)$

3. **a), c)**

4. **c)**

5. **b), c), d)**

6. **a), c), d)**

7. **a), b), c)**

8. **a), b), c), d)**

$$f\left(x\right) = 2^{\{x\}}$$

Clearly f(x) is periodic with period 1.

Now $\int_0^1 2^{\{x\}} dx = \int_0^1 2^x dx = \left[\frac{2^x}{\ln 2}\right]_0^1 = \frac{1}{\ln 2} = \log_2 e$

Also $\int_0^{100} 2^{\{x\}} dx = 100\int_0^1 2^{\{x\}} dx = 100\log_2 e$ $\left[\text{using} \int_0^{na} f(x)dx = n\int_0^a f(x)dx \, if \; a \, is \, the \, period \, of \; f(x)\right]$

PASSAGE TYPE

9. **c)**

10. **b)**

11. **b)**

$f(x) = \sin x + x, f'(x) = \cos x + 1, f''(x) = -\sin x$

f is concave downward for $x \in \left[0, \frac{\pi}{2}\right]$

Irrespective of $k, g(x) = |k|x^2$ is concave upward.

So, If $g\left(\frac{\pi}{2}\right) \le f\left(\frac{\pi}{2}\right)$, then $f(x) \ge g(x)$

$\Rightarrow 1 + \frac{\pi}{2} \ge |k|\frac{\pi^2}{4} \Rightarrow k \in \left[-\frac{(2\pi+4)}{\pi^2}, \frac{(2\pi+4)}{\pi^2}\right]$.

12. **a)**

$f''(x) > 0 \Rightarrow f$ is concave upward.

$\Rightarrow f^{-1}$ is concave downwards.

Consider point dividing the join of this segment in ratio 2:1 is given as

$\left(4, \frac{2f^{-1}(5) + f^{-1}(2)}{3}\right)$, Where upon a point $\left(4, f^{-1}(4)\right)$ on graph of $f^{-1}(x)$ is always above it

$\Rightarrow 3f^{-1}(4) - f^{-1}(2) - 2f^{-1}(5) > 0$.

INTEGER TYPE

1. **0**

Conceptual

2. **0**

Apply property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

3. **1**

Apply Leibnitz rule

4. **9**

$$I = \int_0^1 \frac{dx}{\left(5+2x-2x^2\right)\left(1+e^{2-4x}\right)} \qquad \text{..... (1)}$$

$$= \int_0^1 \frac{dx}{\left[5+2(1-x)-2(1-x)^2\right]\left[1+e^{2-4(1-x)}\right]} \qquad \text{by } \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$= \int_0^1 \frac{dx}{\left(5+2x-2x^2\right)\left(1+e^{-2+4x}\right)}$$

$$= \int_0^1 \frac{e^{2-4x}dx}{\left(5+2x-2x^2\right)\left(1+e^{2-4x}\right)} \qquad \text{.....(2)}$$

(1) + (2)

$$\Rightarrow 2I = \int_0^1 \frac{e^{2-4x}+1}{\left(5+2x-2x^2\right)\left(1+e^{2-4x}\right)} = \int_0^1 \frac{dx}{\left(5+2x-2x^2\right)}$$

$$= -\frac{1}{2}\int_0^1 \frac{dx}{\left(x^2-x-\frac{5}{2}\right)} = -\frac{1}{2}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{11}}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\sqrt{11}}\left[\ln\left(\frac{\frac{\sqrt{11}}{2}+x-\frac{1}{2}}{\frac{\sqrt{11}}{2}-x+\frac{1}{2}}\right)\right]_0^1 = \frac{1}{\sqrt{11}}\ln\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)$$

5. **2**

Conceptual

6. **8**

7. **8**

$f''(x) = f''(5-x)$

$f'(x) = -f'(5-x) + c$

$1 = -7 + C$

C=8

$f'(x) + f'(5-x) = 8$

$$I=\int_1^4 f'(x)dx=\int_1^4 f'(5-x)dx$$

$$2I=\int_1^4 f'(x)dx+\int_1^4 f'(5-x)dx$$

$$=\int_1^4 8\,dx=24$$

I = 12

8. **4**

Here A.M of roots = G.M of roots

So roots should be equal.