LAKSHYA ADVANCED UNIT TEST (LAUT)

00 – 00		Q. Booklet Serial No: 260715	
Test No : 2201	3 Hrs.		

Hints & Solutions

PART C - MATHS

SECTION I - MULTIPLE ANSWER CORRECT TYPE

1. **a)** **1, if n = m**
b) **0, if n > m**
c) **∞, if n < m**

Let P $= \lim_{x\to 0} \frac{\sin x^n}{x^n} . x^n . \frac{x^m}{(\sin x)^m} . \frac{1}{x^m}$

$= 1.(1)^m . \lim_{x\to 0} \frac{x^n}{x^m}$

$= \lim_{x\to 0} \frac{x^n}{x^m}$

Option (a) : If n = m
Then P = 1
Option (b) : If n > m
P $= \lim_{x\to 0} (x)^{n-m} = 0^{n-m} = 0$
Option (c) : If n < m
P $= \lim_{x\to 0} \frac{1}{x^{m-n}} = \frac{1}{0} = \infty$

2. **a)** **continuous $\forall x \in R$**
d) **a periodic function with fundamental period not defined**
$f(x) = 0$ [$\because$ tan $[x]\pi = 0$]
is a constant function.

3. **a)** **f is increasing near a if g(a) > 0**
d) **f is decreasing near a if g(a) < 0**
$f'(x) > 0 \Rightarrow g(x)\,(x - a)^2 > 0$
$\Rightarrow g(x) > 0$
and $f'(x) < 0 \Rightarrow g(x)\,(x - a)^2 < 0$
$\Rightarrow g(x) < 0$.

4. **a)** **$y = 0$**
c) **$y - \frac{4096}{81} = \frac{2048}{27}\left(x - \frac{8}{3}\right)$**

It pass through (2, 0), then
or $0 - y = 4x^3(2 - x)$
or $-x^4 = 4x^3(2 - x)$ ($\because y = x^4$)
$\Rightarrow x^3(8 - 4x + x) = 0$

$\because x = 0, x = \frac{8}{3}$

$\because y = 0, \gamma = \left(\frac{8}{3}\right)^4 = \frac{4096}{81}$

$\because$ Point of tangency are (0, 0)

and $\left(\frac{8}{3}, \frac{4096}{81}\right)$

Tangents are

$y - 0 = 0 \Rightarrow y = 0$ and $y - \frac{4096}{81}$

$= 4\left(\frac{8}{3}\right)^3\left(x - \frac{8}{3}\right)$

i.e., $y = 0$ and $y - \frac{4096}{81} = \frac{2048}{27}\left(x - \frac{8}{3}\right)$

5. **c)** **$-\sqrt{2}$**
d) **$\sqrt{2}$**
$\therefore x = 4t^2 + 3, y = 8t^3 - 1$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{24t^2}{8t} = 3t$

Let the tangent at $P(4t^2 + 3, 8t^3 - 1)$ be normal at

$Q\left(4t_1^2 + 3, 8t_1^3 - 1\right)$

$\therefore$ Equation of tangent at P is

$y - (8t^2 - 1) = 3t\,(x - 4t^2 - 3)$ ...(i)

$\therefore$ Equation (i) passes through Q, then

$8\left(t_1^3 - t^3\right) = 3t.\ 4\left(t_1^2 - t^2\right)$

or $(t_1 - t)^2\,(t + 2t_1) = 0$

$\therefore$ $t = -2t_1$ $(\because\ t \neq t_1)$ ...(ii)

Given, $\left.\frac{dy}{dt}\right|_{\text{at } t} = \frac{1}{\left.\frac{dy}{dx}\right|_{\text{at } t_1}}$

$\Rightarrow 3t = -\frac{1}{3t_1}$

$t\,t_1 = -\frac{1}{9}$

$\Rightarrow -2t_1^2 = -\frac{1}{9}$ ... (iii)

$\therefore$ $t_1 = \pm\frac{1}{3\sqrt{2}}$

$\therefore$ Slope of line $= -\frac{1}{3t_1} = \mp\sqrt{2}$

6. b) sec θ = $\frac{1 + 9t^2}{6t}$

d) sec θ + tan θ = 3t

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - 9t^2}{-6t} = \tan\theta$$

$\therefore$ $\sec\theta = \sqrt{\left(1 + \tan^2\theta\right)}$

$$\sqrt{1 + \frac{\left(1 - 9t^2\right)^2}{36t^2}} = \frac{1 - 9t^2}{6t}$$

and sec θ + tan θ = 3t.

7. a) –30

d) 30

$\because$ $2y^3 = ax^2 + x^3$

$\therefore$ $6y^2\frac{dy}{dx} = 2ax + 3x^2$

$\Rightarrow \frac{dy}{dx} = \left(\frac{2ax + 3x^2}{6y^2}\right)$

$\therefore$ $\left.\frac{dy}{dx}\right|_{(a,a)} = \frac{5a^2}{6a^2} = \frac{5}{6}$

$\Rightarrow$ Equation of tangent is

$y - a = \frac{5}{6}(x - a)$

or $6y - 6a = 5x - 5a$

or $-5x + 6y = a$

or $\frac{x}{(-a/5)} + \frac{y}{(a/6)} = 1$

$\Rightarrow \alpha = -\frac{a}{5},\ \beta = \frac{a}{6}$

$\left(-\frac{a}{5}\right)^2 + \left(\frac{a}{6}\right)^2 = 61$(given)

$\Rightarrow \frac{61a^2}{(30)^2} = 61$

$\therefore$ $a = \pm 30$

8. a) $-2 < b < -1$

d) $1 < b < \infty$

$f(x) = -x^3 + \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2},\ 0 \leq x < 1$

$= 2x - 3,$

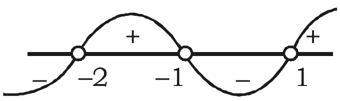
$\therefore$ $f(x)$ has least value at $x = 1$

when $x < 1$, $f(x) > f(1)$

$\therefore$ $\lim_{x \to 1-0} f(x) \geq f(1)$

$$\Rightarrow \; -1^3 + \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \geq -1$$

$$\Rightarrow \; \frac{(b-1)(b^2+1)}{(b+1)(b+2)} \geq 0$$



$\Rightarrow$ $-2 < b < -1$

or $1 \leq b < \infty$

SECTION II – Paragraph Type

9. a) $-\frac{1}{2}$

$\because \lim_{x \to 0^+} f(x)$ = finite

$\Rightarrow \lim_{h \to 0} f(0 + h)$ = finite

$$\Rightarrow \lim_{h \to 0} \frac{\sin h + ae^h + be^{-h} + c \ln(1+h)}{h^3} = \text{finite}$$

at h $\to$ 0, Numerator $\to$ $a + b$ and

Denominator $\to$ 0

$\because$ RHS is finite, $\therefore$ $a + b = 0$

$\Rightarrow$ $b = -a$ … (i)

$$\text{Now, } \lim_{h \to 0} \frac{\sin h + ae^h + ae^{-h} + c \ln(1+h)}{h^3} = \text{finite}$$

$$\Rightarrow \lim_{h \to 0} \frac{\frac{\sin h}{h} + \frac{a(e^h - 1)}{h} - a\frac{(e^{-h} - 1)}{h} + \frac{c \ln(1+h)}{h}}{h^2}$$

= finite

at h $\to$ 0, Numerator $\to$ $1 + a + a + c$ and

Denominator $\to$ 0

$\therefore$ RHS is finite, $\therefore$ $1 + 2a + c = 0$

$\Rightarrow$ $c = -1 - 2a$ … (ii)

$$\therefore \lim_{h \to 0} \frac{\sin h + ae^h - ae^{-h} - (1+2a)\ln(1+h)}{h^3}$$

= finite

(by L' Hospital's rule) $\left(\text{form}\left(\frac{0}{0}\right)\right)$

$$\lim_{h \to 0} \frac{\cos h + ae^h + ae^{-h} - \frac{(1+2a)}{(1+h)}}{3h^2} \quad \left(\text{form}\left(\frac{0}{0}\right)\right)$$

$\therefore$ (by L' Hospital's rule)

$$\lim_{h \to 0} \frac{-\sin h + ae^h - ae^{-h} + \frac{(1+2a)}{(1+h)^2}}{6h}$$

Numerator must be = 0

$$0 + a + a + \frac{1+2a}{(1)^2} = 0$$

$$a = -\frac{1}{2}$$

10. b) $\frac{1}{2}$

11. c) 0

$$\because \; f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f\left(\frac{3x+3h}{3}\right) - f\left(\frac{3x+0}{3}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2 + f(3x) + f(3h)}{3} - \frac{2 + f(3x) + f(0)}{3}}{h}$$

$$= \lim_{h \to 0} \frac{f(3h) - f(0)}{3h - 0} = f'(0)$$

$\Rightarrow$ $f'(2) = f'(0) = 2$ $(\because f'(2) = 2)$

$\Rightarrow$ $f'(x) = 2$

$\Rightarrow$ $f(x) = 2x + c$ …. (i)

Put $x = y = 0$ in $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$

$\Rightarrow$ f(0) = 2

Now, from eq. (i), f(0) = 0 + c = 2

c = 2

From eq. (i),

$f(x) = 2x + 2$

12. c) 4

13. c) $[2, \infty)$

14. c) three non-differentiable points

SECTION III - MATRIX MATCH TYPE

1. A-P,R,T, B-C, C-Q, D-P

(A) $\frac{dy}{dx} = \frac{1}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(t^2,t)} = \frac{1}{2t}$

$\therefore$ Slope of normal = – 2t

$\therefore$ Equation of normal at (t^2,t) is

$y - t = -2t(x - t^2)$

1t passes through (c, 0), then

$0 - t = -2t(c - t^2)$

$\Rightarrow \quad t^2 = c - \frac{1}{2}$

For the distinct values $t^2 > 0$

$\Rightarrow c > \frac{1}{2}$

$\therefore$ c = 1, 5/4, 2(P, R, T)

(B) $\because$ $xy = c^2$

$\therefore$ $x\frac{dy}{dx} + y.\ 1 = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

Subnormal length $= \left|y\frac{dy}{dx}\right|$

$= \left|y\left(\frac{-y}{x}\right)\right|$

$= \frac{y^2}{x}$

$\frac{\frac{y^2}{c^2}}{y} = \frac{y^3}{c} \propto y^3$(S)

(C) $\because$ A + B + C = $\pi \Rightarrow dA + dB + dC = 0$

If R is circumradius,

then $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$\therefore$ $a = 2R \sin A$

or $da = 2R\cos A\, dA$

$\Rightarrow \frac{da}{\cos A} = 2R dA$

$\therefore \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C}$

$= 2R(dA + dB + dc) = 0$ (Q)

2. A-P,Q,R, B-P,Q,R,S,T, C-T, D-P

(A) $\because$ $f(x)$ is continuous in $[0, \pi]$

Then, $f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}} f(x)$

$= \lim_{h\to 0} f\left(\frac{\pi}{4} - h\right)$

$2.\frac{\pi}{4} . \cot\frac{\pi}{4} + b$

$= \lim_{h\to 0} f\left(\frac{\pi}{4} - h\right) + a\sqrt{2}\sin\left(\frac{\pi}{4} - h\right)$

$\Rightarrow \frac{\pi}{2} + b = \frac{\pi}{4} + a$

$\therefore$ $a - b = \frac{\pi}{4}$ ... (i)

and $f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}^+} f(x)$

$= \lim_{h\to 0} f\left(\frac{\pi}{2} + h\right)$

$= \lim_{h\to 0} a\cos(\pi + 2h) - b\sin\left(\frac{\pi}{2} + h\right)$

$0 + b = -a - b$

$\therefore$ $a = -2b$ ... (ii)

From equation (i) and (ii) we, get, $b = \frac{\pi}{12}$

and $a = \frac{\pi}{6}$.

Locus of (a, b) is $x + 2y = 0$ (P, Q, R)

(B) $\because f(x)$ is continuous at $x = 0$

$\therefore$ V.F. $= f(0) = c$

LHL $= \lim\limits_{x\to 0-} f(x)$

$= \lim\limits_{h\to 0} f(0-h)$

$= \lim\limits_{h\to 0} \dfrac{\sin(a+1)(-h)+\sin(-h)}{-h}$

$= \lim\limits_{h\to 0} \dfrac{\sin((a+1)h)(a+1)}{(a+1)h} + \dfrac{\sin h}{h}$

$= a + 1 + 1$

$= a + 2$

RHL $= \lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0} f(0+h)$

$= \lim\limits_{h\to 0} \dfrac{(h+bh^2)^{1/2} - h^{1/2}}{bh^{3/2}}$

$= \lim\limits_{h\to 0} \dfrac{(1+bh)^{1/2} - 1^{1/2}}{1+(bh)-1}, b \neq 0$

$= \dfrac{1}{2}$

$\therefore \quad c = a + 2$

$= \dfrac{1}{2}$

$\Rightarrow \quad c = \dfrac{1}{2}, a = -\dfrac{3}{2}, b \sim R\ \{0\}$

(locus of (a, c) is $x - y + 2 = 0$) (P, Q, R, S, T)

If b is $+3/4$, then $a + 2b = 0$ if b is -1, then $ab = 3/2$

if $b = -\dfrac{3}{2} - \dfrac{\pi}{4}$, then $a - b = \pi/4$

if $b = \dfrac{3}{2} + \dfrac{\pi}{4}$

(C)

SECTION IV - INTEGER TYPE

1. 2

$\int \sin^3 x \cos^5 x\, dx = \int (1 - t^2) t^5\, dt$

for $t = \cos x$

$= \dfrac{t^8}{8} - \dfrac{t^6}{6} + c$

$a = 8$ $b = -6$, $a + b = 2$

2. 5

Given, $f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$

Putting $x = 0$, $y = 0$, we get $f(0) = 0$ ... (i)

and putting $y = -x$, we get

$f(x) + f(-x) = f(0) = 0$

$\therefore \quad f(x) = -f(0)$ ... (ii)

Now, $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(x+h) + f(-x)}{h}$

[from equation. (ii)]

$= \lim\limits_{h\to 0} \dfrac{f\left(\dfrac{x+h-x}{1+(x+h)x}\right)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f\left(\dfrac{h}{1+x^2+xh}\right)}{\left(\dfrac{h}{1+x^2+xh}\right).(1+x^2+xh)}$

$f'(x) = \dfrac{2}{1+x^2}$

$\therefore \quad f'(-2) = \dfrac{2}{1+4} = \dfrac{2}{5}$

and $f(x) = 2\tan^{-1} x + c$

$f(x) = 0 + c = 0$

$\therefore \quad c = 0$

$f(x) = 2\tan^{-1} x$

$\therefore \quad f(\sqrt{3}) = 2\tan^{-1}\sqrt{3} = \dfrac{2\pi}{3}$

3. 4

$\because \quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{6t}{3t^2} = \dfrac{2}{t}$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

$= \dfrac{d}{dx}\left(\dfrac{2}{t}\right)$

$= \dfrac{d}{dt}\left(\dfrac{2}{t}\right).\dfrac{dt}{dx}$

$= \left(-\dfrac{2}{t^2}\right)\left(\dfrac{1}{3t^2}\right)$

$$= \frac{-2}{3t^4}$$

$$\Rightarrow \frac{\left(\frac{d^2y}{dx^2}\right)}{\left(\frac{dy}{dx}\right)^n} = \frac{-\frac{2}{3t^4}}{\left(\frac{2}{t}\right)^n}$$

$$= -\frac{2}{3(2^n)}t^{n-4}$$

For n = 4

4. 2

$y = |x|\, e^{-x}$

$= x\, e^{-x} \quad ; \quad x \ge 0$

$\quad -x\, e^{-x} \quad ; \quad x < 0$

$y' = e^{-x} - xe^{-x} \quad ; \quad x \ge 0$

$\quad -(e^{-x} - xe^{-x}) \quad ; \quad x < 0$

$\therefore$ $f'(x) = 0$ at $x = 1$

$f'(x)$ is not defined at $x = 0$

5. 1

$\because \frac{dy}{dx} = 3ax^2 + 2bx + c$ ... (i)

Since, the curve $y = ax^3 + bx^2 + cx + 5$ touches x-axis at p(–2, 0), then x–axis is the tangent at (–2, 0) the curve meets y-axis in (0, 5).

$\therefore \left.\frac{dy}{dx}\right|_{(0,5)} = 0 + 0 + c = 3$(given) ... (ii)

and $\left.\frac{dy}{dx}\right|_{(-2,0)} = 0 \Rightarrow 12a - 4b + c = 0$

$\Rightarrow 12a - 4b + 3 = 0$ [from equation (ii)] ... (iii)

and (–2, 0) lies on the curve, then

$0 = -8a + 4b - 2c + 5$

$\Rightarrow -8a + 4b - 1 = 0 \quad (\because c = 3)$

$\Rightarrow 8a - 4b + 1 = 0$ ...(iv)

From equation (iii) and (iv) we, get $a = -\frac{1}{2}$,

$b = -\frac{3}{4}$

6. 4

Let $P_1(t_1, t_1^3)$ is a point on the curve $y = x^3$

$\therefore \left.\frac{dy}{dx}\right|_{(t_1, t_1^3)} = 3t_1^2$

Tangent at P_1 is $y - t_1^3 = 3t_1^2(x - t_1)$ ... (i)

The intersection of equation (i) and $y = x^3$

$\Rightarrow x^3 - t_1^3 = 3t_1^2(x - t_1)$

$\Rightarrow (x - t_1)^2(x + 2t_1) = 0$

If $P_2(t_2, t_2^3)$, then $(t_2 - t_1)^2(t_2 + 2t_1) = 0$

$\therefore t_2 = -2t_1 (t_2 \neq t_1)$

Similarly the tangent at P_2 will meet the curve at the point $P_3(t_3, t_3^3)$ when $t_3 = -2t_2 = 4t_1$ and so on. The abscissaes of P_1, P_2, P_3, ... are $t_1, -2t_1, 4t_1, \ldots$ in GP.

$\therefore \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots = -2$(r say)

$\therefore t_2 = t_1 r, t_3 = t_2 r, t_4 = t_3 r$

$$\frac{\text{Area of } \Delta P_1P_2P_3}{\text{Area of } \Delta P_2P_3P_4} = \frac{\frac{1}{2}\left|\begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix}\right|}{\frac{1}{2}\left|\begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix}\right|}$$

$$\frac{\left|\begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix}\right|}{\left|\begin{vmatrix} t_1 r & t_1^3 r^3 & 1 \\ t_2 r & t_2^3 r^3 & 1 \\ t_3 r & t_3^3 r^3 & 1 \end{vmatrix}\right|} = \frac{1}{r^4} = \frac{1}{(-2)^4}$$

$$= \frac{1}{16} = \frac{\lambda}{\mu} \quad \text{(given)}$$

7. 3

Let a be the side of an equilateral trinagle, then

$\frac{da}{dt} = 3$(given)

Now, area of equilateral trinagle A $= \frac{\sqrt{3}}{4} = a^2$

$\therefore dA = \frac{\sqrt{3}}{4} . 2a\frac{da}{dt}$

$$= \frac{\sqrt{3}}{4}.10.3$$

$$= 15\sqrt{3}\ \text{ft}^2/\text{s}$$

$$\therefore \lambda = 15\sqrt{3}$$

8. 4

$$\int \left(x^9 + x^6 + x^3\right)\left(2x^6 + 3x^3 + 6\right)^{1/3} dx$$

$$= \int \left(x^8 + x^5 + x^2\right)\left(2x^9 + 3x^6 + 6x^3\right)^{1/3} dx$$

Let $2x^9 + 3x^6 + 6x^3 = t^3$

$18(x^8 + x^5 + x^2)dx = 3t^2\,dt$

$$= \int \frac{t^2}{6}\, t\, dt = \frac{t^4}{24} + K$$