PART C - MATHS

SECTION I - MULTIPLE ANSWER CORRECT TYPE

1. **b) 55**
   c) 58
   d) 56

$$P = \begin{bmatrix} \omega^2 & \omega^3 & \omega^4 & \ldots & \omega^{n+2} \\ \omega^3 & \omega^4 & \omega^5 & \ldots & \omega^{n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega^{n+2} & \omega^{n+3} & \ldots & \ldots & \omega^{2n+4} \end{bmatrix}$$

or $P^2 =$

$$\begin{bmatrix} \omega^4+\omega^6\ldots & \omega^5+\omega^7+\ldots & \ldots & \ldots \\ \omega^5+\omega^7+\ldots & \ldots & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ldots \\ \omega^{n+4}+\omega^{n+6}\ldots & \ldots & \ldots & \omega^{2n+4}+\omega^{2n+6}\ldots \end{bmatrix}$$

= Null matrix if n is a multiple of 3.

2. **a) $(A(\theta))^{-1} = A(\pi-\theta)$**
   b) $A(\theta) + A(\pi+\theta)$ is a null matrix
   c) $A(\theta)$ is invertible for all $\theta \in R$

We have $|A(\theta)| = 1$

Hence, A is invertible.

$$A(\pi+\theta) = \begin{bmatrix} \sin(\pi+\theta) & i\cos(\pi+\theta) \\ i\cos(\pi+\theta) & \sin(\pi+\theta) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\theta & -i\cos\theta \\ -i\cos\theta & -\sin\theta \end{bmatrix} = -A(\theta)$$

$$\text{adj}(A(\theta)) = \begin{bmatrix} -\sin\theta & -i\cos\theta \\ -i\cos\theta & -\sin\theta \end{bmatrix}$$

$$\Rightarrow (A(\theta))^{-1} = \begin{bmatrix} \sin\theta & -i\cos\theta \\ -i\cos\theta & \sin\theta \end{bmatrix}$$

$$= A(\pi-\theta)$$

3. **b) *a, b, c* are in G.P.**
   d) $(x-\alpha)$ is a factor of $ax^2 + 2bx + c$

Given that

$$\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$$

Operating $C_3 \to C_3 - C_1\alpha - C_2$, we get

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -(a\alpha^2+b\alpha+b\alpha+c) \end{vmatrix} = 0$$

or $(a\alpha^2 + 2b\alpha + c)\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & 1 \end{vmatrix} = 0$

or $(ac - b^2)(a\alpha^2 + 2b\alpha + c) = 0$

$\Rightarrow$ either $ac - b^2 = 0$ or $a\alpha^2 + 2b\alpha + c = 0$

$\Rightarrow$ either *a, b, c* are in G.P or $(x-\alpha)$ is a factor of $ax^2 + 2bx + c$

4. **a) $\frac{2x}{1-x^2}$**
   b) $\tan(2\tan^{-1} x)$
   c) $\tan(\cot^{-1}(-x) - \cot^{-1}(x))$

Let $\tan^{-1} x = \alpha$ and $\tan^{-1} x^3 = \beta$

$\tan\alpha = x$ and $\tan\beta = x^3$

$\therefore \quad 2\tan(\alpha+\beta)$

$$= \frac{2(\tan\alpha + \tan\beta)}{1 - \tan\alpha\tan\beta}$$

$$= 2\left[\frac{x + x^3}{1 - x^4}\right]$$

$$= \frac{2x}{1-x^2}$$

Also $\frac{2x}{1-x^2} = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$= \tan 2\alpha = \tan(2\tan^{-1}x)$

$= \tan(\cot^{-1}(-x) - \cot^{-1}(x))$

5. **b) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$**

   d) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

$PS \times ST = QS \times SR$

Now A.M > G.M

$$\Rightarrow \frac{\frac{1}{PS} + \frac{1}{ST}}{2} > \sqrt{\frac{1}{PS} \times \frac{1}{ST}}$$

$$\Rightarrow \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$$

Also, $\frac{QS + SR}{2} > \sqrt{QS \times SR}$

or $\dfrac{1}{\sqrt{QS \times SR}} > \dfrac{2}{QR}$

or $\dfrac{1}{PS} + \dfrac{1}{ST} > \dfrac{4}{QR}$

6. **a)** $\boldsymbol{a = 2}$

c) $\mathbf{L = \dfrac{1}{64}}$

$$L = \lim_{x\to 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}, a > 0$$

$$= \lim_{x\to 0} \frac{a - a\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} - \frac{x^2}{4}}{x^4}$$

$$= \lim_{x\to 0} \frac{a - a\left[1 - \frac{x^2}{2a^2} - \frac{1}{8}\frac{x^4}{a^4} + ...\right] - \frac{x^2}{4}}{x^4}$$

$$= \operatorname{Lt}_{x\to 0} \frac{\left(\frac{x^2}{2a} - \frac{x}{4}\right) + \left(\frac{x^4}{8a^3} + ...\right)}{x^4}$$

The above expression is $\dfrac{\text{polynomial}}{\text{polynomial}}$ form.

$\therefore$ $2a = 4 \Rightarrow a = 2$

$L = \dfrac{1}{8a^3} = \dfrac{1}{64}$

7. **b)** **cot $\boldsymbol{x}$ + cosec $\boldsymbol{x}$**

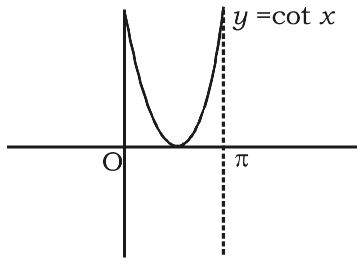
c) $\begin{cases} \mathbf{3 - 2\sin x}, & \mathbf{0 < x \le \frac{\pi}{2}} \\ \mathbf{x + 1 - \left(\frac{\pi}{2}\right)}, & \mathbf{\frac{\pi}{2} < x < \pi} \end{cases}$

On $(0, \pi)$

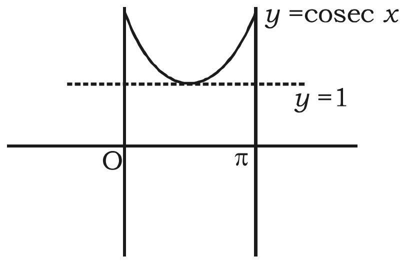
i) $\tan x = f(x)$

we know that $\tan x$ is discontinuous at $x = \pi/2$

ii) $\cot x \rightarrow$



cosec $x \rightarrow$



$\therefore$ $\cot x + \operatorname{cosec} x$ is continuous

iii) $f(x) = \begin{cases} 3 - 2\sin x & 0 < x \le \frac{\pi}{2} \\ x + 1 - \left(\frac{\pi}{2}\right), & \frac{\pi}{2} < x < \pi \end{cases}$

LHL= $\operatorname{Lt} f(x) = 3 - 2\sin x$

$x \Rightarrow \frac{\pi}{2}^{-}$ $x \Rightarrow \frac{\pi}{2}^{-}$

$= 1$

RHL $= \operatorname{Lt} f(x) = \operatorname{Lt}\left(x + 1 - \frac{\pi}{2}\right)$

$x \Rightarrow \frac{\pi}{2}^{+} = x \Rightarrow \frac{\pi}{2}^{+}$

$= 1$

At $x = \dfrac{\pi}{2}$

$f(x) = 3 - 2\sin\dfrac{\pi}{2} = 1$

LHS $= f(x) =$ RHS at $x = \dfrac{\pi}{2}$

$\therefore$ $f(x)$ is continuous in the given domain.

iv) $f(x) = \begin{cases} x\sin x, & 0 < x \le \pi/2 \\ \frac{\pi}{2}\sin(\pi + x), & \frac{\pi}{2} < x < \pi \end{cases}$

Here, $f(x)$ will be continuous on $(0, \pi)$ if it is continuous at $x = \pi/2$. At $x = \pi/2$

$$\text{L.H.L} = \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)$$

$$= \lim_{h\to 0} f\left(\frac{\pi}{2}-h\right)\sin\left(\frac{\pi}{2}-h\right)$$

$$= \frac{\pi}{2}\sin\frac{\pi}{2} = \frac{\pi}{2}$$

$$\text{R.H.L} = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$$

$$= \lim_{h\to 0} \frac{\pi}{2}\sin\left(\pi+\frac{\pi}{2}+h\right)$$

$$= \frac{\pi}{2}\sin\left(\pi+\frac{\pi}{2}\right)$$

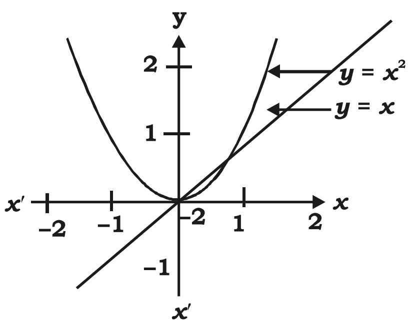
$$= \frac{-\pi}{2}\sin\frac{\pi}{2} = -\frac{\pi}{2}$$

As L.H.L $\neq$ R. H.L., $f(x)$ is not continuous.

8. **a) h is continuous for all x**
   b) h is differentiable for all x
   d) h is not differentiable at two values of x

From the figure it is clear that

$$h(x) = \begin{cases} x, \text{if} & x \le 0 \\ x^2, \text{if} & 0 < x < 1 \\ x, \text{if} & x \ge 1 \end{cases}$$



From the graph, it is clear that h(x) is continuous for all $x \in$ R, h′(x) = 1, and h is not differentiable at x = 0 and 1

9.

SECTION II - PARAGRAPH TYPE

9. **b) [–5, 5]**
   c) [2, 3]

10. **b) one real, negative root**
    c) two imaginary roots

$$f(x) = \begin{cases} \dfrac{a(1-x\sin x)+b\cos x+5}{x^2}, & x<0 \\ 3, & x=0 \\ \left\{1+\left(\dfrac{P(x)}{x}\right)\right\}^{1/x}, & x>0 \end{cases}$$

where $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$f(0) = 3$.

R.H.L. = $\lim_{x\to 0+} f(x)$

$= \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$

$$= \lim_{h\to 0}\left\{1+\left(\frac{P(h)}{h}\right)\right\}^{1/h}$$

since f is continuous at x = 0, R.H.L. exists. For the existance of R.H.L., a_0, a_1 = 0. Thus,

$$\text{R.H.L.} = \lim_{h\to 0}\left(1+a_2h+a_3h^2\right)^{1/h} \quad (1^\infty\text{form})$$

$$= e^{\lim_{h\to 0}(1+a_2h+a_3h^2)1/h} = e^{a_2}$$

$$\text{L.H.L.} = \lim_{x\to 0} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0}\frac{a(1-(-h)\sin(-h))+b\cos(-h)+5}{(-h)^2}$$

$$= \lim_{h\to 0}\frac{a(1-h(h))+b\left(1-\frac{h^2}{2!}\right)+5}{h^2}$$

For finite value of L.H.L., $a + b + 5 = 0$ and $-a-\frac{b}{2} = 3$.

Solving, we get $a = -1$, $b = -4$.

Now, g(x) = 3a sin x –b cos x = –3 sin x + 4 cos x

which has range [–5, 5]. Also, $P(x) = a_3x^3 + (\log_e 3)x^2$

$P''(x) = 6a_3x + 2\log_e 3$

$\therefore$ $P''(0) = 2\log_e 3$

Further, $P(x) = b$ or $a_3x^3 + (\log_e 3)x^2 = 4$ has only one real root, as the graph of $P(x) = a_3x^3 + (\log_e 3)x^2$ meets $y = -4$ only once for negative value of x.

11. **a) R = 2.5**
b) r = 1

12. **b) smallest angle is $\sin^{-1}\left(\frac{3}{5}\right)$**

c) The triangle is right angled

We have,

$$r_1 = \frac{\Delta}{s-a} = 2, r_2 = \frac{\Delta}{s-b} = 3, r_3 = \frac{\Delta}{s-c} = 6$$

Given $\Delta = 6$,

$\therefore$ $s - a = 3$ (i)

$s - b = 2$ (ii)

$s - c = 1$ (iii)

Adding Eqs. (i) and (ii), $2s - a - b = 5$ or $a + b + c - a - b = 5$ or $c = 5$

Adding Eqs. (i) and (iii), $2s - a - c = 4$, $\therefore b = 4$ and adding Eqs. (ii) and (iii), $2s - b - c = 3$, $\therefore a = 3$

Hence the sides of the Δ are $a = 3, b = 4, c = 5$.

Since the triangle is right angled, the greatest angle is 90^0. Also, the least angle is opposite to side a, which is

$\sin^{-1}\frac{3}{5}$. Therefore,

$$90^0 - \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{3}{5}$$

Also $R = \frac{abc}{4\Delta} = \frac{60}{24} = 2.5$

$$r = \frac{\Delta}{a} = \frac{6}{6} = 1$$

SECTION III - INTEGER TYPE

1. **3**

Equation $x^3 + ax^2 + bx + c = 0$ has roots α, β, γ. Therefore,

$\alpha + \beta + \gamma = -a$

$\alpha\beta + \alpha\gamma + \gamma\alpha = b$

Since the given system of equations has non- trivial solutions, we have

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$$

or $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = 0$

or $(\alpha + \beta + \gamma)[\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha] = 0$

or $(\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha) = 0$

$\Rightarrow -a[a^2 - 3b] = 0$ or $a^2/b = 3$

2. **8**

In a skew symmetric matrix, diagonal elements are zero.

Also $a_{ij} + a_{ji} = 0$

Hence, number of matrices = 2 × 2 × 2 = 8

3. **9**

Let M = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow b = -1$, e = 2, h = 3

$$M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\Rightarrow a = 0$, d = 3, g = 2

$$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

$\Rightarrow$ g + h + i = 12

$\Rightarrow$ i = 7

Therefore, sum of diagonal elements = 9

4. **1**

$$\lim_{x\to\infty}\left(f(x) + \frac{3f(x)-1}{f^2(x)}\right) = 3$$

or $\left(\lim_{x\to\infty} f(x) + \frac{3\lim_{x\to\infty} f(x) - 1}{\left(\lim_{x\to\infty} f(x)\right)^2}\right) = 3$

or $\left(y + \frac{3y-1}{y^2}\right) = 3$

or $y^3 - 3y^2 + 3y - 1 = 0$

or $(y-1)^3 = 0$

or $y = 1$

5. **8**

In $(f(x))$ = In $(x-1)$ + In $(x-2)$ + ... In $(x-n)$

or $f'(x) = f(x)\left[\frac{1}{x-1}+\frac{1}{x+2}+\ldots\frac{1}{x-n}\right]$

$= (x-2)(x-3)(x-n) + (x-1)(x-3)\ldots(x-n) + \ldots + (x-1)(x-2)\ldots(x-(n-1))$

$f'(n) = (n-1)(n-2)(n-3).3.2.1$

(all other factors except the last vanishes when $x = n$) or 5040 = $(n-1)!$

or n = 8

6. **4**

As f(x) is continuous in [1, 10], f(x) will attain all values between f(1) and f(10). As f(x) takes rational values for all x and there are innumerable irrational values between f(1) and f(10), it implies that f(x) can take rational values for all 'x' if f(x) has a constant value at all points between x = 1 and x = 10. Given that f(2) = 4, Then f(3) = 4

7. **1**

$f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

$= \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{2\cos^2\theta - 1}}\right)\right)$

$= \sin(\sin^{-1}(\tan\theta) = \tan\theta$

$\therefore \frac{d(\tan\theta)}{d(\tan\theta)} = 1$

8. **3**

$2\cos\frac{\pi}{2k} + 2\cos\frac{\pi}{k} = \sqrt{3}+1$

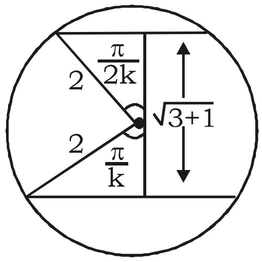
or $\cos\frac{\pi}{2k} + \cos\frac{\pi}{k} = \frac{\sqrt{3}+1}{2}$

Let $\frac{\pi}{k} = \theta$. Then,

$\cos\theta + \cos\frac{\theta}{2} = \frac{\sqrt{3}+1}{2}$

or $2\cos^2\frac{\theta}{2} - 1 + \cos\frac{\theta}{2} = \frac{\sqrt{3}+1}{2}$

or $2t^2 + t - \frac{\sqrt{3}+3}{2} = 0$ [where $\cos(\theta/2) = t$]



or $t = \frac{-1 \pm \sqrt{1+4\left(3+\sqrt{3}\right)}}{4}$