LAKSHYA ADVANCED UNIT TEST (LAUT)

00 – 00		MATHS PAPER - II	
Test No : -----	3 Hrs.		

Hints & Solutions

SECTION I - MULTIPLE ANSWER CORRECT

1. **a)** $a_1 + |a_2| = 1$
c) $a_2 + |a_3| = 1$
d) $|a_2| + a_3 = 3$

a) $\lim\limits_{x\to 0^+} f(x) = 1$

c) $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2 = 1$

2. **b)** $f(x) = \tan(\tan^{-1} x)$ & $g(x) = \cot(\cot^{1} x)$
c) $f(x) = \text{sgn}(x)$ & $g(x) = \text{sgn}(\text{sgn}(x))$
d) $f(x) = \cot^2 x.\cos^2 x$ & $g(x) = \cot^2 x - \cos^2 x$

3. **b)** $14 f(1) = f(3)$
c) $9 f(3) = 2 f(5)$
[Hint : $f(x) = 1 + x^n$ or $1 - x^n \Rightarrow n = 3$
$\Rightarrow f(x) = x^3 + 1$]

4. **b)** $f(x) = \cos\left(\frac{|\sin x|}{x}\right)$

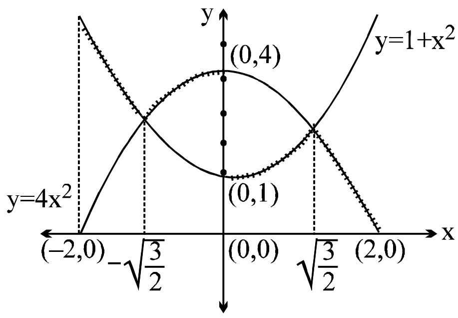
c) $f(x) = x \sin\frac{\pi}{x}$

d) $f(x) = \frac{1}{\ln|x|}$

5. **b)** **has a point of discontinuity**
d) **is not differentiable at more than one point**

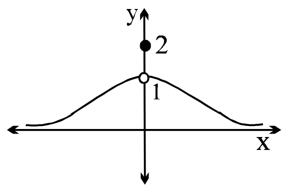
[Hint : $f(x)$ max. $(4-x^2, 1+x^2)$, $-2 \le x \le 0$

min. $(4 - x^2, 1+x^2)$, $0 < x \le 2$]



6. **d)** $(0, 1) \cup \{2\}$

[Sol.



$$\left[\frac{1}{\ln(x^2+e)}\right] - \begin{cases} 0 & x \ne 0 \\ 1 & x = 0 \end{cases}$$

$$f(x) - \begin{cases} 2 & x = 0 \\ \frac{1}{\sqrt{1+x^2}} & x \ne 0 \end{cases}$$

Hence range of $f(x)$ is $(0, 1) \cup \{2\}$]

7.

8.

SECTION II - Paragraph Type

9. **d)** $16 + f(0)$

$$f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2$$

$\therefore\quad f\left(\frac{x}{2}\right)-2f\left(\frac{x}{4}\right)+f\left(\frac{x}{8}\right)=\left(\frac{x}{2}\right)^2$

$f\left(\frac{x}{4}\right)-2f\left(\frac{x}{8}\right)+f\left(\frac{x}{16}\right)=\left(\frac{x}{4}\right)^2$

$f\left(\frac{x}{2^n}\right)-2f\left(\frac{x}{2^{n+1}}\right)+f\left(\frac{x}{2^{n+2}}\right)=\left(\frac{x}{2^n}\right)^2$

Adding, we get

$f(x)-f\left(\frac{x}{2}\right)-f\left(\frac{x}{2^{n+1}}\right)+f\left(\frac{x}{2^{n+2}}\right)=$

$x^2\left(1+\frac{1}{2^2}+\ldots+\frac{1}{2^{2n}}\right)$

As $n\to\infty$, we get $f(x)-f\left(\frac{x}{2}\right)=\frac{4x^2}{3}$

Repeating the same procedure again, we get

$f(x)-f(0)=\frac{16x^2}{9}$

$f(3)=16+f(0)$

10. c) two solutions

Clearly, the two solution

11. b) $-\frac{1}{2}$

$\lim_{x\to0^+}\frac{\ln f(x)}{\ln g(x)}=\lim_{x\to0}\frac{-x/2}{x}=-\frac{1}{2}$

12. d) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$h(x)=\tan^{-1}(\ln(\ln 1/x^2))0<x<1$

$1<\frac{1}{x^2}<\infty\Rightarrow 0<\ln\frac{1}{x^2}<\infty$

$\therefore\quad -\infty<\ln(\ln(1/x^2))<\infty$

$\therefore$ Range of $h(x)$ is $(-\pi/2, \pi/2)$.

SECTION III - Integer Type

1. 2

$f(x-1)+f(x+1)=\sqrt{3}f(x)$

$\Rightarrow f(x)+f(x+2)=\sqrt{3}f(x+1)$

Putting $x = x + 2$

$f(x+1)+f(x+3)=\sqrt{3}f(x+2)$

$f(x-1)+2f(x+2)+f(x+3)=\sqrt{3}[\sqrt{3}f(x+1)]$

$f(x-1)+f(x+3)=f(x+1)$

Putting $x = x + 2$ again $f(x+1)+f(x+5)=f(x+3)$

$f(x-1)+f(x+5)=0$

$f(x+5)=-f(x-1)$

$f(x)=-f(x+6)$

$f(x+12)=f(x)$

$\sum_{r=0}^{19}f(5+12r)=20f(5)=20\times10=200$

Hence, sum of digit = 2 + 0 + 0 = 2

2. 1

Put $x = 1$ in $2f(x)=f(xy)+f\left(\frac{x}{y}\right)$ ...(i)

$2f(1)=f(y)+f\left(\frac{1}{y}\right)$ ...(ii)

Replace x by y and y by x in Eq. (i), we get

$2f(y)=f(yx)+f\left(\frac{y}{x}\right)$ ...(iii)

From Eqs. (i) and (iii),

$2\{f(x)-f(y)\}=f\left(\frac{x}{y}\right)-\left\{-f\left(\frac{x}{y}\right)\right\}=2f\left(\frac{x}{y}\right)$ ...(iv)

$f(x)-f(y)=f\left(\frac{x}{y}\right)$ ...(v)

$\lim_{h\to0}\frac{f(1+h)-f(1)}{h}=f'(1)=1$

$\lim_{h\to0}\frac{f(1+h)}{h}=1$ as $f(1)=0$

$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f\left(1+\frac{h}{x}\right)}{h}=\frac{1}{x}$

$f(x)=\log|x|+c$

$f(1)=0\Rightarrow c=0$

$f(e)=1$

3. 3

$f'(0^+)=\lim_{h\to0}\frac{h^m\sin\frac{1}{x}}{h}$ must exist $\Rightarrow m>1$

for $m > 1$, $h'(x) =$

$$\begin{cases} mx^{m-1}\sin\frac{1}{x} - x^{m-2}\cos\frac{1}{x}, & \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}$$

Now, $\lim_{h\to 0} h(x) = \lim_{h\to 0} mh^{m-1}\sin\frac{1}{h} - h^{m-2}\cos\frac{1}{h}$

limit exist if $m > 2$

$\therefore \quad m \in \mathbb{N} \Rightarrow m = 3$

4. 3

$$\lim_{h\to 0}\frac{f(2+2h+h^2)-f(2)}{(2h+h^2)}$$

$$\lim_{h\to 0}\frac{(h^2+h)}{f(1+h^2+h)-f(1)}\cdot\lim_{h\to 0}\frac{2h+h^2}{h^2+h}$$

$$= \frac{f'(2)}{f'(1)}\cdot 2 = \frac{6\times 2}{4} = 3 \text{ (By L' Hopital's rule)}$$

5. 1

$$\lim_{x\to\infty}\frac{\cot^{-1}\left(\frac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\frac{a^x}{\log_a x}\right)}$$

$$\lim_{x\to\infty}\left(\frac{\log_a x}{x^a}\right)\to 0 \text{ and } \left(\frac{a^x}{\log_a x}\right)\to\infty$$

(using L' Hopital's rule)

$\therefore \quad l = \frac{\pi/2}{\pi/2} = 1$

6. 1

$l = \lim_{x\to 1}(\cos^{-1} x)^{1-x}$ $\quad$ (0^0 form)

$\text{In } l = \lim_{x\to 1}(1-x)\text{In}(\cos^{-1}x)$

Put $\quad x = \cos\theta$

but $\lim_{\theta\to 0}(1-\cos\theta)\text{In}\theta = \lim_{\theta\to 0}\frac{\text{In}\theta}{\frac{1}{(1-\cos\theta)}} =$

$$\lim_{\theta\to 0}\frac{(1-\cos\theta)^2}{\sin\theta\cdot\theta}$$

$$\lim_{\theta\to 0}\frac{(1-\cos\theta)^2}{\theta^2\frac{\sin\theta}{\theta}} = 0 \Rightarrow l = 1$$

7. 2

$\tan^2 x$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sec^2 x$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow$ Number of discontinuities = 2

8. 0