LAKSHYA ADVANCED UNIT TEST (LAUT)

00 – 00		MATHS - PAPAER I	
Test No : -----	3 Hrs.		

Hints & Solutions

SECTION I - MULTIPLE ANSWER CORRECT

1. **a)** $a_1 + |a_2| = 1$

c) $a_2 + |a_3| = 1$

d) $|a_2| + a_3 = 3$

$\lim\limits_{x\to 0}\left(2+\dfrac{f(x)}{x^2}\right)^{1/x}$ exists only when

$$\lim_{x\to 0}\left(2+\frac{f(x)}{x^2}\right) = 1$$

$$\Rightarrow \lim_{x\to 0}\left(2+\frac{a_0+a_1x+a_2x^2+a_3x^3+....}{x^2}\right) = 1$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{a_0+a_1x+a_2x^2+a_3x^3+....}{x^2}\right) = -1$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{a_0}{x^2}+\frac{a_1}{x}+a_2+a_3x+....\right) = -1$$

Since RHS is finite and equal to –1.

$\therefore$ In LHS, a_0 and a_1 must be equal to zero.

$\therefore$ $a_0 = 0.,\ a_1 = 0$

Now from (i),

$\lim\limits_{x\to 0}(a_2 + a_3x + ...) = -1 \quad \Rightarrow \quad a_2 = -1$

$\therefore$ $f(x) = -x^2 + a_3x^3 + \ldots$

Given , $\lim\limits_{x\to 0}\left(2+\dfrac{f(x)}{x^2}\right)^{1/x} = e^2$

$\Rightarrow \lim\limits_{x\to 0}(2 - 1 + a_3x + ...)^{1/x} = e^2$

$\Rightarrow \lim\limits_{x\to 0}(1 + a_3x + ...)^{1/x} = e^2$

$\Rightarrow e^{\lim\limits_{x\to 0}(a_3x+....)\times\frac{1}{x}} = e^2$

$\Rightarrow e^{a_3} = e^2$

$\therefore$ $a_3 = 2$

Hence, $a_0 = 0,\ a_1 = 0,\ a_2 = -1,\ a_3 = 2$

Alternate (a) : $a_1 + |a_2| = 0 + 1 = 1$

Alternate (b) : $|a_1| + a_2 = 0 - 1 = -1$

Alternate (c) : $a_2 + |a_3| = -1 + 2 = 1$

Alternate (d) : $|a_2| + a_3 = 1 + 2 = 3$

2. **b)** is zero for $0 < a < b$

c) is non-existance for $a > b > 0$

d) id $e^{-(1/a)}$ or $e^{-(1/b)}$ for $a = b$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{ax+1}{bx+2}\right)^x$$

$$= \lim_{x\to\infty}\left(\frac{a+\frac{1}{x}}{b+\frac{2}{x}}\right)^x \quad .... \text{(i)}$$

If $0 < \dfrac{a}{b} < 1$, then $\lim\limits_{x\to\infty} f(x) = 0$

$\therefore$ $\lim\limits_{x\to\infty} f(x) = 0$ for $0 < a < b$

If $\dfrac{a}{b} > 1$, then $\lim\limits_{x\to\infty} f(x)$ does not exist

$\therefore$ $\lim\limits_{x\to\infty} f(x)$ is non-existant for $a > b > 0$

For $a = b$, from (i)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{ax+1}{bx+2}\right)^x$$

$$= \lim_{x\to\infty}\left(1-\frac{1}{ax+2}\right)^x \quad (1^\infty \text{form})$$

$$= e^{\lim\limits_{x\to\infty}\frac{-x}{ax+2}} = e^{-\lim\limits_{x\to\infty}\frac{1}{a+2/x}} \quad (\because a = b)$$

$= e^{-1/a}$ or $e^{-1/b}$

3. **a)** f is continuous at $x = 0$ but discontinuous at $x = 1$

b) f is continuous at $x = 1$ but discontinuous at $x = 0$

c) f is continuous at $x = 1$ and $x = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{n\to\infty}\left\{\lim_{x\to 0^-}(\cos^2 x)^n\right\}$$

$$= \lim_{n\to\infty}\left\{\lim_{h\to 0}(\cos^2 h)^n\right\}$$

$= 0 \quad (\because 0 \le \cos^2 h < 1)$

and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{n\to\infty}\left\{\lim\limits_{x\to 0^+}(1 + x^n)^{1/n}\right\}$

$= \lim_{n\to\infty}\left\{\lim_{h\to 0}(1+h^n)^{1/n}\right\} = 1$

Also $f(0) = 1 \Rightarrow f$ is discontinuous at $x = 0$

and $f(1^-) = \lim_{x\to 1^-} f(x) = \lim_{h\to 0^+} f(1-h)$

$= \lim_{h\to 0^+}\left(\lim_{n\to\infty}(1+(1-h)^n)^{1/n}\right)$

$= \lim_{n\to\infty}\left(\lim_{h\to 0^+}(1+(1-h)^n)^{1/n}\right)$

$= \lim_{n\to\infty}(1)^{1/n} = 1^0 = 1$

and $\lim_{x\to 1^+} f(x) = \lim_{h\to 0^+} f(1+h)$

$= \lim_{h\to 0^+}\left(\lim_{n\to\infty}\frac{1}{1+(1+h)^n}\right)$

$= \lim_{n\to\infty}\lim_{h\to 0^+}\left(\frac{1}{1+(1+h)^n}\right)$

$= 0$

and $f(1) = 1$

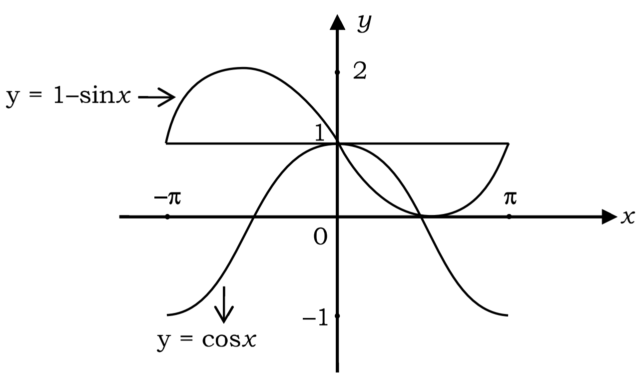
$\Rightarrow$ f is discontinuous at $x = 1$

4. **a)** $f(x)$ is not differentiable at $x = 0$

c) $f(x)$ is continuous at $x = 0$

d) $f(x)$ is non differentiable at $x = \pi/2$

$f(x) = \min\{1, \cos x, 1-\sin x\}$ $-\pi \le x \le \pi$



$$f(x)\begin{cases}\cos x & [-\pi, 0] \\ 1-\sin x & \left(0, \frac{\pi}{2}\right] \\ \cos x & \left(\frac{\pi}{2}, \pi\right]\end{cases}$$

f is non differentiable at $x = 0$ & $\frac{\pi}{2}$

5. **a)** $f(x) = |x| \sin x$

b) $g(x) = \tan(\tan^{-1}x)$

c) $h(x) = \ln e^x$

d) $l(x) = \lim_{n\to\infty}\frac{2|x|}{\pi}\tan^{-1}(nx)$

The function $f : R \to R$ is called an identity function if $f(x) = x \ \forall \ x \in R$

Alternate (a) :

$$\text{sgn}\, x = \begin{cases}\frac{|x|}{x} \text{ or } \frac{x}{|x|} & , x \ne 0 \\ 0 & , x = 0\end{cases}$$

$$|x|\,\text{sgn}\, x = \begin{cases} x , & x \ne 0 \\ 0 , & x = 0\end{cases}$$

$f(x) = x \ \forall \ x \in R$

Alternate (b) : $g(x) = \tan(\tan^{-1}x) = x \ \forall \ x \in R$

Alternate (c) : $h(x) = \ln e^x \ \forall\, x \in R = x \ln e \ \forall\, x \in R$

$= x \ \forall\, x \in R$

Alternate (d) : $l(x) = \lim_{h\to\infty}\frac{2|x|}{\pi}\tan^{-1}(nx)$

For $x > 0$

$l(x) = \lim_{n\to\infty}\frac{2x}{\pi}\tan^{-1}(nx) = \frac{2x}{\pi}\tan^{-1}(\infty)$

$= \frac{2x}{\pi}\cdot\frac{\pi}{2} = x$

For $x = 0$

$l(x) = \lim_{n\to\infty}\tan^{-1}(0) = 0$

and for $x < 0$

$l(x) = \lim_{n\to\infty}\frac{2(-x)}{\pi}\tan^{-1}(nx)$

$= -\frac{2x}{\pi}\tan^{-1}(-\infty) = -\frac{2x}{\pi}\times\frac{-\pi}{2} = x$

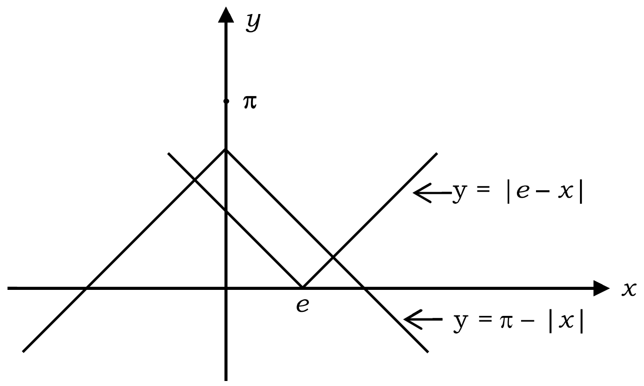
Hence, $l(x) = x \ \forall \ x \in R$

6. **b)** $f(x)$ is many one function

c) $f(x)$ is not a periodic function

d) $f(x)$ is neither even nor odd

$f(x) = \min(|e-x|, \pi - |x|)$



$$e - x = \pi + x$$

$$x = \frac{e-\pi}{2}$$

$$y = (e-x) \Rightarrow \left|e - \left(\frac{e-\pi}{2}\right)\right| = \frac{e+\pi}{2}$$

$$R_f \in \left(-\infty, \frac{e+\pi}{2}\right)$$

many one function
non periodic function
neither even nor odd function

7. b) 0

$$\sin^{-1}\left\{\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right)\right\}$$

$$\Rightarrow \sin^{-1}\left\{\cot\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sec^{-1}\left(\sqrt{2}\right)\right)\right\}$$

$$\Rightarrow \sin^{-1}\left\{\cot\left(\frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4}\right)\right\}$$

$$\Rightarrow \sin^{-1}\left\{\cot\left(\frac{\pi + 2\pi + 3\pi}{12}\right)\right\}$$

$$\Rightarrow \sin^{-1}\left\{\cot\left(\frac{\pi}{2}\right)\right\}$$

$$\Rightarrow \sin^{-1}(0) = 0$$

8. a) $2\tan^{-1} x = -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ for $x < -1$

b) $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $x \geq 0$

c) $2\tan^{-1} x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ for $x > 1$

d) $\tan^{-1} x = -\pi + \cot^{-1}\left(\frac{1}{x}\right)$ for $x < 0$

(a) $2\tan^{-1}x = -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; $-\sin^{-1}\left(\frac{2x}{1+x^2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; $-\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right) \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right]$

$$-\frac{3\pi}{2} \leq 2\tan^{-1} x \leq -\frac{\pi}{2}$$

$$-\frac{3\pi}{4} \leq \tan^{-1} x \leq -\frac{\pi}{4}$$

$$\Rightarrow x \leq -1$$

(b) $2\tan^{-1} x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, where $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \in [0, \pi]$

$$0 \leq 2\tan^{-1} x \leq \pi$$

$$0 \leq \tan^{-1} x \leq \frac{\pi}{2}$$

$$x \in [0, \infty)$$

(c) $2\tan^{-1} x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\tan^{-1}\left(\frac{2x}{1-x^2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; $\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$\frac{\pi}{2} < 2\tan^{-1} x < \frac{3\pi}{2}$$

$$\frac{\pi}{4} < \tan^{-1} x < \frac{3\pi}{4} \Rightarrow x > 1$$

9. b) 2011

d) 2013

It is clear that $f(x)$ and $f^{-1}(x)$ are symmetrical about the line

$$y = x$$

$$\Rightarrow f(x) = x \quad (\because y = f(x))$$

$$\Rightarrow x^2 - 2ax + a(a+1) = x$$

$$\Rightarrow (x-a)2 - (x-a) = 0$$

or $(x-a)(x-a-1) = 0$

$\therefore \quad x = a$ or $a + 1$

If $a = 2012$, then $a + 1 = 2013$

and if $a + 1 = 2012$, then $a = 2011$

10. a) function is continuous everywhere

b) function is differentiable everywhere

d) function is periodic

$$f(x) = \begin{cases} \left[|x|\left[\frac{1}{|x|}\right]\right] & |x| \neq \frac{1}{n} \\ 0 & |x| = \frac{1}{n} \end{cases}$$

$$|x| > 1 \Rightarrow 0 < \frac{1}{|x|} < 1$$

$$\Rightarrow \left[\frac{1}{|x|}\right] = 0$$

$$\Rightarrow \quad \left[|x|\left[\frac{1}{|x|}\right]\right] = 0$$

$$0 < |x| < 1 \Rightarrow \quad n < \frac{1}{|x|} < n + 1$$

$$\Rightarrow \quad \left[\frac{1}{|x|}\right] = n \qquad - \text{(i)}$$

$$\frac{1}{n+1} < |x| < \frac{1}{n} \qquad - \text{(ii)}$$

$$\text{(i), (ii)} \quad \Rightarrow \quad \frac{n}{n+1} < |x| \times \left[\frac{1}{|x|}\right] < \frac{n}{n}$$

$$\left[|x| \times \left[\frac{1}{|x|}\right]\right] = 0$$

$\Rightarrow f(x) = 0 \qquad \forall \ x \in R$

Continuous everywhere

Differentiable everywhere

Periodic

SECTION II - MATRIX MATCH

1. A- P,Q,R,S ; B- P,Q,R ; C-Q,S ; D-S

(A) $\because \ f(x) = e^{\ln[1+\{x\}]}$

$= [1 + \{x\}] \qquad (\because e^{\ln x} = x)$

$= 1 \qquad (\because 0 < \{x\} < 1)$

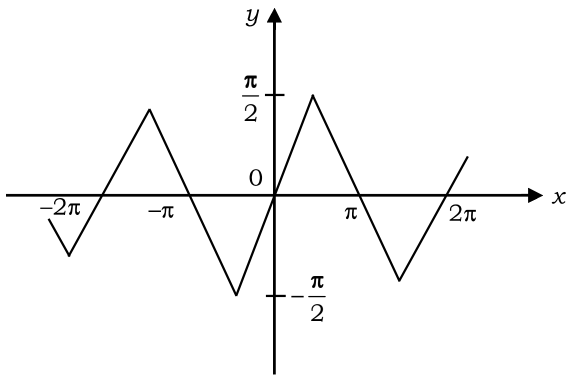
Here $D_f = R$, $\quad R_f = \{1\}$

and $f(x)$ is periodic & symmetrical about y-axis.

(B) $\because \ g(x) = \sin^{-1}(\sin x)$

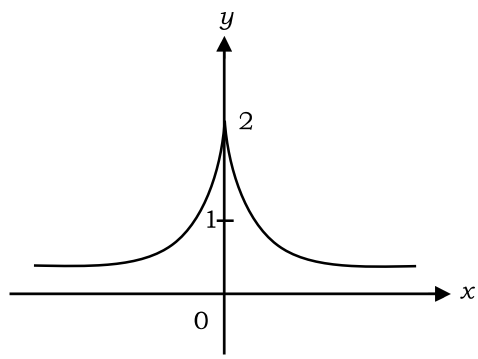
Here $D_g = R$, Range contains of only one natural number.

and $g(x)$ is periodic and not symmetric about y-axis.



(C) $\because \ h(x) = 2e^{-|x|}$

Hence $D_h = R$, and $h(x)$ is non-periodic and symmetrical about y-axis



(D) $\quad k(x) = \tan^{-1}\sqrt{[x]+[-x]} + \sqrt{3-|x|} + \frac{1}{x^2}$

$\therefore \ k(-x) = \tan^{-1}\sqrt{[-x]+[x]} + \sqrt{3-|-x|} + \frac{1}{(-x)^2}$

$= \tan^{-1}\sqrt{[x]+[-x]} + \sqrt{3-|x|} + \frac{1}{x^2} = k(x)$

$\therefore \ k(x)$ is symmetric about y-axis.

and $k(x)$ is non-periodic.

The function $k(x)$ is defined only when

$[x] + [-x] \geq 0$, $3 - |x| \geq 0 \quad \& \quad x \neq 0$

$\Rightarrow$ $x \in I$, $-3 \le x \le 3$ & $x \ne 0$

$\therefore$ $D_k = \{-3, -2, -1, 1, 2, 3\} \ne R$

at one value of x, $k(x)$

12. A - P,R ; B - P,Q,T ; C - P,Q,R,T ; D - P,S

Since $\frac{\sin x}{x} \to 1$ as $x \to 0$ but less than 1.

$\therefore$ $\frac{n_1 \sin x}{x} \to n_1$ as $x \to 0$ but less than n_1.

Similarly,

$\frac{x}{\sin x} \to 1$ as $x \to 0$ but more than or equal to 1.

$\therefore$ $\frac{n_2 x}{\sin x} \to n_2$ as $x \to 0$ but more than or equal to n_2.

$\therefore$ $L = \lim_{x \to 0}\left[\frac{n_1 \sin x}{x}\right] = n_1 - 1$ and

$M = \lim_{x \to 0}\left[\frac{n_2 x}{\sin x}\right] = n_2$

Also $\frac{\tan x}{x} \to 1$ as $x \to 0$ but more than 1.

$\therefore$ $\frac{n_3 \tan x}{x} \to n_3$ as $x \to 0$ but more than or equal to n_3.

Similarly,

$\frac{x}{\tan x} \to 1$ as $x \to 0$ but less than 1.

$\therefore$ $\frac{n_4 x}{\tan x} \to n_4$ as $x \to 0$ but less than n_4.

$\Rightarrow$ $N = \lim_{x \to 0}\left[\frac{n_3 \tan x}{x}\right] = n_3$ and

$P = \lim_{x \to 0}\left[\frac{n_4 x}{\tan x}\right] = n_4 - 1$

(A) For $n_1 = 3$, $n_4 = 9$

$\therefore$ $\frac{P}{L} = \frac{n_4 - 1}{n_1 - 1} = \frac{9-1}{3-1} = 4$ (P, R)

(B) For $n_2 = 12$, $n_3 = 2$

$\therefore$ $\frac{M}{N} = \frac{n_2}{n_3} = \frac{12}{2} = 6$ (P, Q, T)

(C) For $n_2 = 3$, $n_4 = 10$

$\therefore$ $P + M = n_4 - 1 + n_2 = 10 - 1 + 3 = 12$

(P, Q, R, T)

(D) For $n_1 = 4$, $n_2 = 5$, $n_3 = 6$, $n_4 = 7$

$\therefore$ $L + N + P - M = n_1 - 1 + n_3 + n_4 - 1 - n_2$

$= 4 - 1 + 6 + 7 - 1 - 5$

$= 10$ (P, S)

SECTION III - INTEGER TYPE

1. 2

We have $\lim_{x \to 0} \frac{1 + a\cos 2x + b\cos 4x}{x^4} = c$

$\Rightarrow$

$$\lim_{x \to 0} \frac{1 + a\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right) + b\left(1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} + \ldots\right)}{x^4}$$

$= c$

$\Rightarrow$

$$\lim_{x \to 0} \frac{(1 + a + b)\frac{x^2}{2!}(-4a - 16b) + \frac{x^4}{4!}(16a + 256b) + \frac{x^6}{6!}(-64a - 4096b) + \ldots}{x^4}$$

$= c$

$$\Rightarrow \lim_{x \to 0}\left(\frac{1 + a + b}{x^4}\right) + \frac{(-4a - 16b)}{2!\,x^2} + \frac{(16a + 16b)}{2!\,x^2} + \frac{x^2}{6!}(-64a - 4096b) + \ldots = c \quad \ldots(i)$$

Since RHS is finite

$\therefore$ $1 + a + b = 0$ and $-4a - 16b = 0$

$\Rightarrow$ $a = -4/3$, $b = 1/3$

Now from (i),

$$\frac{16 \times -\frac{4}{3} + 256 \times \frac{1}{3}}{4!} = c \quad \Rightarrow \quad c = \frac{8}{3}$$

Hence, $[a^{-1} + b^{-1} + c^{-1}] = \left[-\frac{3}{4} + \frac{3}{1} + \frac{3}{8}\right] = \left[\frac{21}{8}\right] = 2$

2. 8

Since the given limit exists and denominator approaches zero as $x \to 0$, then numerator must approach zero as $x \to 0$, which is possible if $4 + B = 0 \Rightarrow B = -4$ which is obtained by substituting zero in place of x in numerator.

$$L = \lim_{x\to 0} \frac{4+\sin 2x + A\sin x - 4\cos x}{x^2} \quad \left(\frac{0}{0}\text{form}\right)$$

$$L = \lim_{x\to 0} \frac{2\cos 2x + A\cos x + 4\sin x}{2x}$$

(By L' Hopital's Rule)

Again denominator $\to 0$,

which numerator $\to 2 + A$

Hence for above limit to exist, A + 2 = 0 $\Rightarrow$ A = –2

$$\therefore \quad L = \lim_{x\to 0} \frac{2\cos 2x - 2\cos x + 4\sin x}{2x} \left(\frac{0}{0}\text{form}\right)$$

$$L = \lim_{x\to 0} \frac{-4\sin 2x + 2\sin x + 4\cos x}{2}$$

(By L' Hopital's Rule)

$$= \frac{0+0+4}{2} = 2$$

Hence, AB = (–2) (–4) = 8

3. 9

$$\text{LHL} = \lim_{x\to 2^-} g(x) = \lim_{h\to 0} g(2-h)$$

$$= \lim_{h\to 0}\left(1+\frac{15(2-h-2)}{(2-h)^2+1}\right)^{\frac{1+\ln 3}{\tan(2-h-2)}}$$

$$= \lim_{h\to 0}\left(1-\frac{15h}{(2-h)^2+1}\right)^{\frac{\ln 3e}{\tan h}}$$

$$= e^{\lim_{h\to 0}\frac{\ln 3e}{\tan h}\times\frac{15h}{(2-h)^2+1}}$$

$$= e^{\lim_{h\to 0}\frac{h}{\tan h}\cdot\frac{15\ln 3e}{(2-h)^2+1}}$$

$$= e^{1\cdot\frac{15\ln 3e}{4+1}}$$

$$= e^{3\ln 3e} = e^{\ln(3e)^3} = 27e^3$$

$$\text{RHL} = \lim_{x\to 2^+} g(x) = \lim_{h\to 0} g(2+h) =$$

$$k\lim_{h\to 0}\left(\frac{e^{f(2+h)}-2e^3}{\sin(2+h-2)}\right)$$

$$= k\lim_{h\to 0}\left(\frac{e^{f(2+h)}-2e^3}{\sin h}\right)$$

$$= k\lim_{h\to 0}\left(\frac{e^{\ln(2+h)+2+h+1}-2e^3}{\sin h}\right)$$

$$= ke^3\lim_{h\to 0}\left(\frac{e^{\ln(2+h)+h}-2}{\sin h}\right)$$

$$= ke^3\lim_{h\to 0}\left(\frac{e^{\ln(2+h)+h}\cdot\left(\frac{1}{2+h}+1\right)-0}{\cos h}\right)$$

(by L' Hopital's Rule)

$$= ke^3\left(\frac{e^{\ln 2}\cdot\left(\frac{1}{2}+1\right)}{1}\right) = 3ke^3$$

$\because$ $g(x)$ is continuous in [1, 3].

$27e^3 = 27e^3 = 3ke^3$

$\therefore$ $k = 9$

4. 3

Let $f(x) = e^{-|x|}$

$$\therefore \quad Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{e^{-|0-h|}-e^{-0}}{-h}$$

$$= \lim_{h\to 0}\frac{e^{-h}-1}{-h} = 1$$

given $$\lim_{h\to 0}\frac{1-\cos\left(1-\cos\frac{x}{2}\right)}{2^\lambda x^\mu} = 1$$

$$\Rightarrow \frac{1}{2^\lambda}\lim_{x\to 0}\frac{1-\cos\left(1-\cos\frac{x}{2}\right)}{\left(1-\cos\frac{x}{2}\right)^2}\cdot\left(\frac{1-\cos\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)^2\cdot\frac{x^4}{2^4\cdot x^\mu} = 1$$

$$\Rightarrow \frac{1}{2^\lambda}\cdot\lim_{x\to 0}\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)^2\cdot\frac{1}{x^{\mu-4}\cdot 2^4} = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{1}{x^{\mu-4}\cdot 2^{\lambda+7}} = 1$$

Since RHS is finite non-zero quantity.

$\therefore$ $\mu - 4 = 0 \Rightarrow \mu = 4$

and $2^{\lambda+7} = 1 = 2^0 \Rightarrow \lambda + 7 = 0$

$\Rightarrow$ $\lambda = -7$

Hence, $|\lambda| - |\mu| = 7 - 4 = 3$

5. 4

$f(x) + 2f\left(\frac{2002}{x}\right) = 3x$

$x = 1001$

$f(1001) + 2f(2) = 3 \times 1001$ ...(i)

$x = 2$

$f(2) + 2f(1001) = 3 \times 2$ ...(ii)

By solving (i) and (ii)

$-3f(1001) = 3003 - 12$

$f(1001) + 1001 = 4$

6. 5

$\because \quad x \in (\sqrt{2}, \sqrt{3})$

$\Rightarrow \quad \sqrt{2} < x < \sqrt{3}$ or $\frac{1}{\sqrt{3}} < \frac{1}{x} < \frac{1}{\sqrt{2}}$

$\therefore \quad \left\{\frac{1}{x}\right\} = \frac{1}{x}$ and $2 < x^2 < 3$

or $0 < x^2 - 2 < 1$

$\therefore \quad \{x^2\} = x^2 - 2$

Given $\left\{\frac{1}{x}\right\} = \{x^2\} \Rightarrow \frac{1}{x} = x^2 - 2$ ... (i)

$\Rightarrow \quad x^3 - 2x - 1 = 0$

$\Rightarrow \quad (x + 1)(x^2 - x - 1) = 0$

$x \neq -1 \quad (\because -1 \notin (\sqrt{2}, \sqrt{3}))$

$\therefore \quad x^2 - x - 1 = 0 \Rightarrow x^2 = x + 1$

$\Rightarrow \quad x^4 = x^2 + 2x + 1 = x + 1 + 2x + 1$

$(\because x^2 = x + 1)$

$\therefore \quad x^4 = 3x + 2$ ... (ii)

$\Rightarrow \quad x^4 - \frac{3}{x} = (3x + 2) - 3(x^2 - 2)$

(from eqs. (i) and (ii))

$= (3x + 2) - 3(x + 1 - 2)$

$(\because x^2 = x + 1)$

$= 5$

7. 5

$f(x) = 3|\sin x| - 2|\cos x|$

Period = π

$$f(x) = \begin{cases} 3\sin x - 2\cos x & \left[0, \frac{\pi}{2}\right) \\ 3\sin x + 2\cos x & \left[\frac{\pi}{2}, \pi\right) \end{cases}$$

$$f'(x) = \begin{cases} 3\cos x + 2\sin x & \left[0, \frac{\pi}{2}\right) \\ 3\cos x - 2\sin x & \left[\frac{\pi}{2}, \pi\right) \end{cases}$$

$$f'(x) = \begin{cases} > 0 & \left[0, \frac{\pi}{2}\right) \\ < 0 & \left[\frac{\pi}{2}, \pi\right) \end{cases}$$

$f(0) = -2$ and $f(\pi) = -2$

$f\left(\frac{\pi}{2}\right) = 3$

$R_f = [-2, 3]$

$|b - a| = 5$

8. 2

Here,

$\tan^{-1} x + \cos^{-1}\frac{y}{\sqrt{1-y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$

$\Leftrightarrow \quad \tan^{-1} x + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}(3)$

$\Leftrightarrow \quad \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1} 3 - \tan^{-1}(x)$

$\Leftrightarrow \quad \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{3-x}{1+3x}\right)$

$\Leftrightarrow \quad y = \frac{1+3x}{3-x}$

As x, y are positive integers, $x = 1, 2$ and corresponding $y = 2, 7$.

$\therefore$ Solutions are $(x, y) = (1, 2), (2, 7)$ (i.e.) two solutions.