LAKSHYA ADVANCED UNIT TEST (LAUT)

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Test No : -----	3 Hrs.		

Hints & Solutions

PART C - MATH

SECTION I - MULTIPLE ANSWER CORRECT TYPE

1. **b)** $f(x) = \log_x e = \dfrac{1}{\log_e x} = g(x)$

   Hence, the functions are identical.

   c) $f(x) = \sin(\cos^{-1}x) = \sin\left(\dfrac{\pi}{2} - \sin^{-1} x\right) =$

   $\cos(\sin^{-1}x) = g(x)$

   Hence, the functions are identical.

   d) $\sin^2 x + \cos^2 x$ and $\text{sgn}(x^2 + 1)$

   $\sin^2 x + \cos^2 x = \text{sgn}\,(x^2 + 1)$

2. **a, b, c.**

   $f(x) + f(y) = f\left(\dfrac{x+y}{1-xy}\right)$

   Replacing y by $-x$, we get $f(x) + f(-x) = f(0)$ (1)

   Putting $x = y = 0$, we get $f(0) + f(0) = f(0)$ or $f(0) = f(0)$

   $\therefore f(x) + f(-x) = 0$ [From (1)]

   Hence, $f(x)$ is an odd function.

   $f(x) + f(y) = f\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

   Replacing y by $-x$, we get $f(x) + f(-x) = f(0)$ (2)

   Putting $x = y = 0$, we get $f(0) + f(0) = f(0)$ or $f(0) = f(0)$

   $\therefore f(x) + f(-x) = 0$ [From (2)]

   Hence, $f(x)$ is an odd function.

   $f(x + y) = f(x) + f(y)$

   Replacing y by $-x$, we get $f(x) + f(-x) = f(0)$ (3)

   Putting $x = y = 0$, we get $f(0) + f(0) = f(0)$ or $f(0) = f(0)$

   $\therefore f(x) + f(-x) = 0$ [From (3)]

   Hence, $f(x)$ is an odd function.

3. **b, d.**

   $f\left(x + \dfrac{1}{x}\right) = x^2 + \dfrac{1}{x^2}$

   or $f\left(x + \dfrac{1}{x}\right) = x^2 + \dfrac{1}{x^2} = \left(x + \dfrac{1}{x}\right)^2 - 2$

   or $f(y) = y^2 - 2$

   Now, $y = x + \dfrac{1}{x} \geq 2$ or ≤ -2

   Hence, the domain of the function is $(-\infty, -2] \cup [2, \infty)$.

   Also, for these value of y, $y^2 \geq 4$, or $y^2 - 2 \geq 2$.

   Hence, the range of the function is $[2, \infty)$.

4. **a, b, c, d.**

   $f(x + 1) = \dfrac{f(x) - 5}{f(x) - 3}$ (1)

   or $f(x)f(x + 1) - 3f(x + 1) = f(x) - 5$

   or $f(x) = \dfrac{3f(x+1) - 5}{f(x+1) - 1}$

   Replacing x by $(x - 1)$, we get

   $f(x - 1) = \dfrac{3f(x) - 5}{f(x) - 1}$ (2)

   Using (1), $f(x + 2) = \dfrac{f(x+1) - 5}{f(x+1) - 3} = \dfrac{\dfrac{f(x)-5}{f(x)-3} - 5}{\dfrac{f(x)-5}{f(x)-3} - 3}$

   $= \dfrac{2f(x) - 5}{f(x) - 2}$ (3)

   Using (1), $f(x - 2) = \dfrac{3f(x-1) - 5}{f(x-1) - 1}$

   $= \dfrac{3\left(\dfrac{3f(x)-5}{f(x)-1}\right) - 5}{\dfrac{3f(x)-5}{f(x)-1} - 1}$

$= \dfrac{2f(x)-5}{f(x)-2}$ (4)

Using (3) and (4), we have $f(x + 2) = f(x - 2)$.

Therefore, $f(x + 4) = f(x)$, i.e. $f(x)$ is periodic with period 4.

5. **a, d.**
$f(x) = \sec^{-1}[1 + \cos^2 x]$
$f(x)$ is defined if $[1 + \cos^2 x] \le -1$ or $[1 + \cos^2 x] \ge 1$
i.e., $[\cos^2 x] \le -2$ (not possible) or $[\cos^2 x] \ge 0$
i.e., $\cos^2 \ge 0$ or $x \in R$
Now, $0 \le \cos^2 x \le 1$ or $1 \le 1 + \cos^2 x \le 2$
or $[1 + \cos^2 x] = 1,2$
or $\sec^{-1}[1 + \cos^2 x] = \sec^{-1}1, \sec^{-1}2$
Hence, the range is $\{\sec^{-1}1, \sec^{-1}2\}$.

6. **a, b, c.**

f(x) is defined if $\log_{|\sin x|}(x^2 - 8x + 23) - \dfrac{3}{\log_2 |\sin x|}$ > 0

or $\log_{|\sin x|}\left(\dfrac{x^2 - 8x + 23}{8}\right) > 0$

This is true if $|\sin x| \ne 0, 1$ and $\dfrac{x^2 - 8x + 23}{8} < 1$

Now, $\dfrac{x^2 - 8x + 23}{8} < 1$ or $x^2 - 8x + 15 < 0$

or $x \in (3, 5) - \left\{\pi + \dfrac{3\pi}{2}\right\}$

Domain = $(3, \pi) \cup \left(\pi, \dfrac{3}{2}\right) \cup \left(\dfrac{3\pi}{2}, 5\right)$.

7. **b, c.**
As (0, 0) and $(x, g(x))$ are two vertices of an equilateral triangle, length of the side of Δ is

$\sqrt{(x-0)^2 + (g(x)-0)^2} = \sqrt{x^2 + (g(x))^2}$

$\therefore$ Area of equilateral $(\Delta) = \dfrac{\sqrt{3}}{4}(x^2 + (g(x))^2)$

$= \dfrac{\sqrt{3}}{4}$

or $g(x)^2 = 1 - x^2$

or $g(x) = \pm\sqrt{1 - x^2}$

Therefore, (b) and (c) are the correct answers as (a) is not a function (since image of x is not unique).

8. **a, c.**
$f(x) = \cos[\pi^2]\, x + \cos[-\pi^2]x$
We know $9 < \pi^2 < 10$ and $-10 < -\pi^2 < -9$
or $[\pi^2] = 9$ and $[-\pi^2] = -10$
or $f(x) = \cos 9x + \cos(-10x)$
$= \cos 9x + \cos 10x$

a. $f\left(\dfrac{\pi}{2}\right) = \cos\dfrac{9\pi}{2} + \cos 5\pi = -1$ (true)

c. $f(-\pi) = \cos(-9\pi) + \cos(-10\pi) = \cos 9\pi + \cos 10\pi$
$= -1 + 1 = 0$ (true)

SECTION II - PARAGRAPH TYPE

$f(x) = \begin{cases} x+1, & x \le 1 \\ 2x+1, & 1 < x \le 2 \end{cases}$

$g(x) = \begin{cases} x^2, & -1 \le x < 2 \\ x+2, & 2 \le x \le 3 \end{cases}$

$\therefore f(x) = \begin{cases} g(x)+1, & g(x) \le 1 \\ 2g(x)+1, & 1 < g(x) \le 2 \end{cases}$

or $f(g(x)) = \begin{cases} x^2+1, & x^2 \le 1, & -1 \le x < 2 \\ x+2+1, & x+2 \le 1, & 2 \le x \le 3 \\ 2x^2+1, & 1 < x^2 \le 2, & -1 \le x < 2 \\ 2(x+2)+1, & 1 < x+2 \le 2, & 2 \le x \le 3 \end{cases}$

or $f(g(x)) = \begin{cases} x^2+1, & -1 \le x < 1 \\ 2x^2+1, & 1 < x \le \sqrt{2} \end{cases}$

9. **c)**
Hence, the domain of $f(x)$ is $[-1, \sqrt{2}\,]$.

10. **c)**
For $-1 \le x \le 1$, we have $x^2 \in [0, 1]$ or $x^2 + 1 \in [1, 2]$.
For $1 < x \le \sqrt{2}$, we have $x^2 \in (1, 2]$ or $2x^2 + 1 \in (3, 5]$.
Hence, the range is $[1, 2] \cup (3,5]$.

11.

$(f(x))^2 f\left(\dfrac{1-x}{1+x}\right) = 64x$ (1)

Putting $\dfrac{1-x}{1+x} = y$ or $x = \dfrac{1-y}{1+y}$, we get

$$\left\{\left(f\,\frac{1-y}{1+y}\right)\right\}^{2} \cdot f(y) = 64\left(\frac{1-y}{1+y}\right)$$

or $f(x) \cdot \left\{f\left(\frac{1-x}{1+x}\right)\right\}^{2} = 64\left(\frac{1-x}{1+x}\right)$ (2)

Squaring (1) and dividing by (2),

$$\frac{f(x)^4\left\{f\left(\frac{1-x}{1+x}\right)\right\}^2}{f(x)\left\{f\left(\frac{1-x}{1+x}\right)\right\}^2} = \frac{(64x^2)}{64\left(\frac{1-x}{1+x}\right)}$$

or $\{f(x)\}^3 = 64x^2\left(\frac{1+x}{1-x}\right)$

or $f(x) \quad = 4x^{2/3}\left(\frac{1+x}{1-x}\right)^{1/3}$

or $x = f(9/7) = -4(9/7)^{2/3}(2)$

SECTION III - INTEGER TYPE

1. **1**

$\left|\left|\left|x^2 - x + 4\right| - 2\right| - 3\right| = x^2 + x - 12$

or $\left|\left|x^2 - x + 2\right| - 3\right| = x^2 + x - 12$

or $\left|x^2 - x - 1\right| = x^2 + x - 12$

or $2x = 11$

or $x = 11/2$

2. **5**

$x! - (x-1)! \neq 0$ or $x \in I^{+1} - \{1\}$

$2^{\left(\frac{\pi}{\tan^{-1}x}\right)} > 4$ as $\tan^{-1}x < \frac{\pi}{2}$

or $\frac{(x-4)(x-10)}{(x-1)!(x-1)} < 0$

or $x \in \{5, 6, \ldots, 9\}$

(The domain of the function is only positive integers)

3. **5**

As a > 2,

$a^2 > 2a > a > 2$

Now, $(x-a)(x-2a)(x-a^2) < 0$

Thus, the solution set is as shown.

Between 0, a, there are $(a-1)$ positive integers and between $2a$, a^2, there are $a^2 - 2a - 1$ integers. Therefore,

$a^2 - 2a - 1 + a - 1 = 18$ or $a^2 - a - 20 = 0$

or $(a-5)(a+4) = 0$

$\therefore a = 5$

4. **2**

$f(x) + f\left(\frac{1}{x}\right) + f(x) = \frac{1}{x^2} + x$

or $\frac{1}{x^2} + x = x^2 + \frac{1}{x}$

or $x - \frac{1}{x} = x^2 - \frac{1}{x^2}$

or $\left(x - \frac{1}{x}\right) = \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)$

or $\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x} - 1\right) = 0$

$x = \frac{1}{x}$; $x + \frac{1}{x} = 1$ (rejected)

Hence, $x = 1$ or -1.

5. **0**

Let $x = \frac{|a|}{a} + \frac{|b|}{b} + \frac{|c|}{c}$

If exactly one –ve, then $x = 1$.

If exactly two –ve, then $x = -1$.

If all three –ve, then $x = -3$.

If all three +ve, then $x = 3$.

Then the required sum is 0.

6. **6**

$\because k \in$ odd

$f(k) = k + 3$

$f(f(k)) = \frac{k+3}{2}$

If $\frac{k+3}{2}$ is odd, then $27 = \frac{k+3}{2} + 3$ or $k = 45$ which is not possible.

Therefore, $\frac{k+3}{2}$ is even. Thus,

$$27 = f(f(f(k))) = f\left(\frac{k+3}{2}\right) = \frac{k+3}{4}$$

$\therefore\ k = 105$

Verifying, $f\,(f\,(f\,(105))) = f\,(f(108)) = f(54) = 27$

$\therefore\ k = 105$

7. **3**

Clearly, fundamental period is $\frac{4\pi}{3}$. Then z lies in the third quadratnt.

8. **7**

$$f(x) = \frac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x}$$

$$= \underbrace{ax^7 + bx^5 + cx^3 + dx + \frac{1}{x}}_{\textbf{odd function}} + 15$$

Now, $f(x) + f(-x) = 30$

or $f(x) + f(-x) = 30$

or $f(-5) = 30 - f(5) = 28$